LECTURE 4: SPAN AND $Ax = 0$

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I- SPAN

Last time, we discovered the notion of linear combinations, which is a neat way of combining vectors

Def: A linear combo of u, v, w is an expression of the form

au + bv + cw (a, b, c are numbers)

Example:

$$2\begin{bmatrix}1\\2\end{bmatrix} + (-3)\begin{bmatrix}3\\4\end{bmatrix} + (1)\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$au \quad + bv \quad + cw$$

Now, you may ask: What if we take ALL the linear combinations of u, v, w? This has a special name, called the SPAN:

DEF SPAN {u, v, w} = Set of ALL linear combos of u, v, w

Warning: {u,v,w} only has 3 elements (u, v, and w), but Span{u,v,w} is HUGE usually infinite; it contains 2u, v+w, 3u+4v+5w, etc.

Ex Is $\begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$ IN $Span \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$?

Are there a, b, c such that:

$$a \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + c \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix} \text{ ?}$$

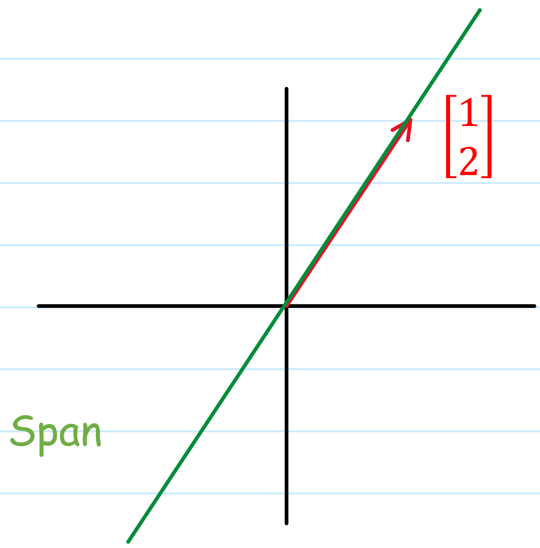
Is $\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{array}\right]$ consistent ?

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

Inconsistent, so NO

Note: Geometrically, the Span looks like a line or a plane, or all of R^m

Ex: Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ = Line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

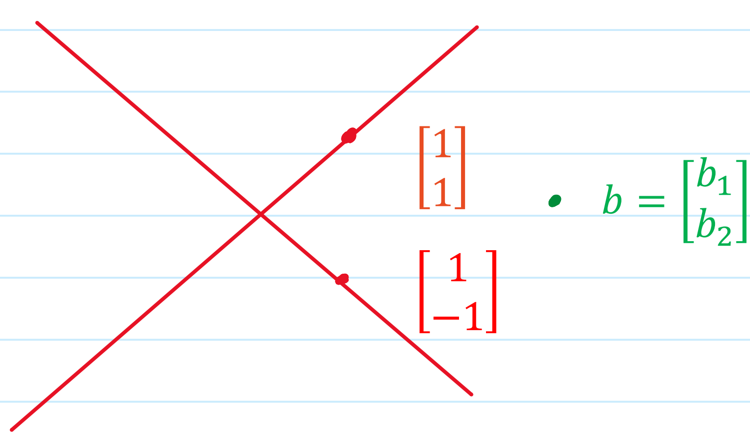


II- THE ROW THEOREM

Note: The Span may or may not be all of R^m (like in the example above), so an important question to ask is:

When is Span = R^m ?

Example: Is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$?



Stupid Way: Show any $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is a linear combo of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Smart Way: Consider:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \; (\times -1) \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

Pivot in every row, so YES

(Why? $Ax = b$ is consistent for any b, by last time)

In fact, this is so useful, let me isolate it as a separate theorem:

<u>THE ROW THEOREM:</u>

A has a pivot in every <u>ROW</u>

$\Leftrightarrow$ $Ax = b$ is consistent for <u>EVERY</u> b

$\Leftrightarrow$ Span of the columns of A is R^m

ROW

CONSISTENT

SPAN

(They all go together)

Example: Is

$$\text{Span}\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix}\right\} = \mathbb{R}^3 \; ?$$

Span $\left\{\begin{bmatrix}1\\-4\\-3\end{bmatrix},\begin{bmatrix}3\\2\\-2\end{bmatrix},\begin{bmatrix}4\\-6\\-7\end{bmatrix}\right\} = \mathbb{R}^3$?

$$\begin{bmatrix}1 & 3 & 4\\-4 & 2 & -6\\-3 & -2 & -7\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix}1 & 3 & 4\\0 & 14 & 0\\0 & 0 & 0\end{bmatrix}$$

NOT a pivot in every row, so NO

Note: Think of Span{u,v,w} as the "information" expressed by u,v,w.

Example: Span{EGG,WATER,SALT} = all recipes with egg, water, salt (So OMELETTE is in it, but CAKE is not in it)

III- THE EQUATION Ax = 0

Now let's move on an discuss a special case of Ax = b, namely the case where b is 0, that is let's look at Ax = 0
Turns out to be as important as the original equation

Some Facts:

1) Ax = 0 is always consistent

(Why? x = 0 is a solution, since A0 = 0)

2) Ax = 0 either has 1 solution or infinitely many solutions

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(Why? If $x \neq 0$ solves Ax = 0, then so does 2x, since A(2x) = 2Ax = 2(0) = 0, and so does 3x, 4x, etc.)

Example: Solve Ax = 0, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 4 & 8 & 12 & 16 \\ 6 & 12 & 18 & 24 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 8 & 12 & 16 & 0 \\ 6 & 12 & 18 & 24 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} & & z & t & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Backsubstitution:

$$\begin{cases} x + z = 0 \\ y + z + 2t = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ y = -z - 2t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -z \\ -z - 2t \\ z \\ t \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -2t \\ 0 \\ t \end{bmatrix}$$

$$= z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

IV- $Ax = b$ vs. $Ax = 0$

Question: How are $Ax = 0$ and $Ax = b$ related?

Fun Fact: The general solution of $Ax = b$ is

$$x = x_0 + x_p$$

Where x_0 is the general solution to $Ax = 0$ and x_p is a particular solution to $Ax = b$

Example: Solve $Ax = b$, where A is as above and

$$b = \begin{bmatrix} 0 \\ -2 \\ -8 \\ -12 \end{bmatrix}$$

(Note: Inappropriate for the exam, because in general you would just to it directly)

By Fact:

$x = x_0 + x_p$

$$= \underbrace{z\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{}$$

From above, solves $Ax = 0$

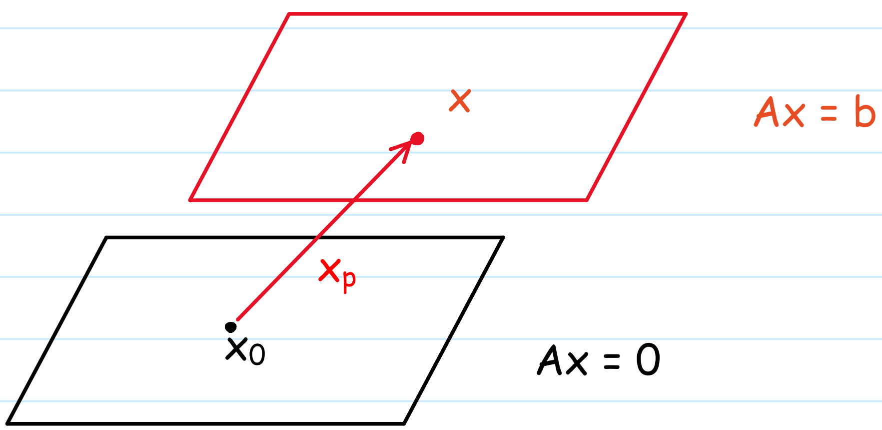
ONE solution of $Ax = b$ (Pure guessing)

What this says is: Once you solved $Ax = 0$, you basically found all the solutions to $Ax = b$ (as long as you find one solution)

This is why $Ax = 0$ is so important, it lies at the heart of solving $Ax = b$. In fact, this is why sometimes $Ax = 0$ is called the kernel (nucleus) of A, and we'll study this more later.

Geometric interpretation:

$x = x_0 + x_p$ means that you literally translate x_0 by the vector x_p to get x



In other words, geometrically, $Ax = b$ is just a translate $Ax = 0$. So if $Ax = 0$ is a plane, then $Ax = b$ (if it exists) is a plane as well

Remarks:

1) One of the consequences is that $Ax = b$ must have 0, 1, or infinitely many solutions. It cannot have exactly two solutions.

2) Main Question (next time): When does $Ax = 0$ imply $x = 0$?