LECTURE 4: THE TRANSPORT EQUATION

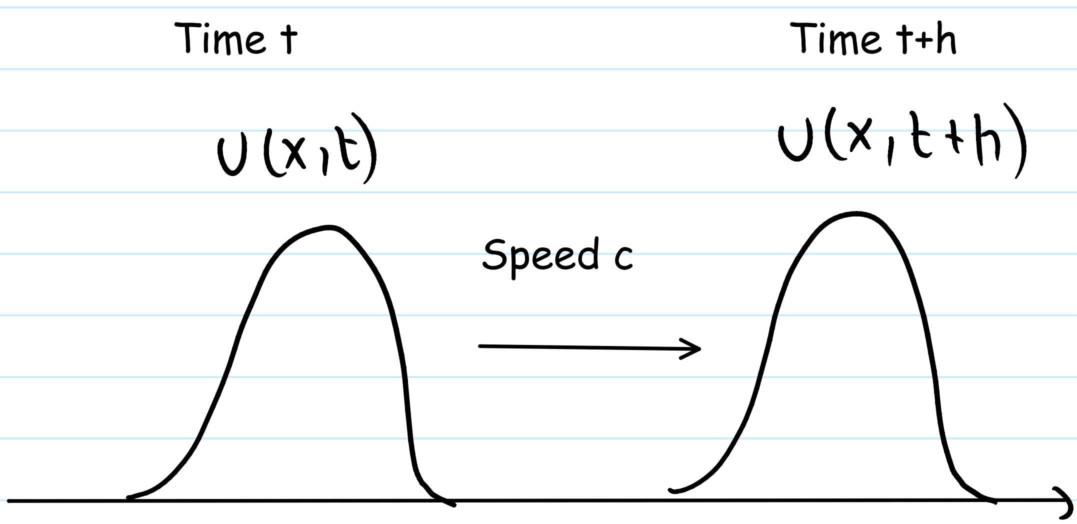
Thursday, October 3, 2019 6:40 PM

Today: I would like to discuss an important example of a first-order linear PDE that has a physical significance: The Transport Equation

Transport Equation: $u = u(x,t)$, c = constant

$$\boxed{u_t + c\,u_x = 0}$$

Intuitively: Represents density of fluid when it is transported at a speed c

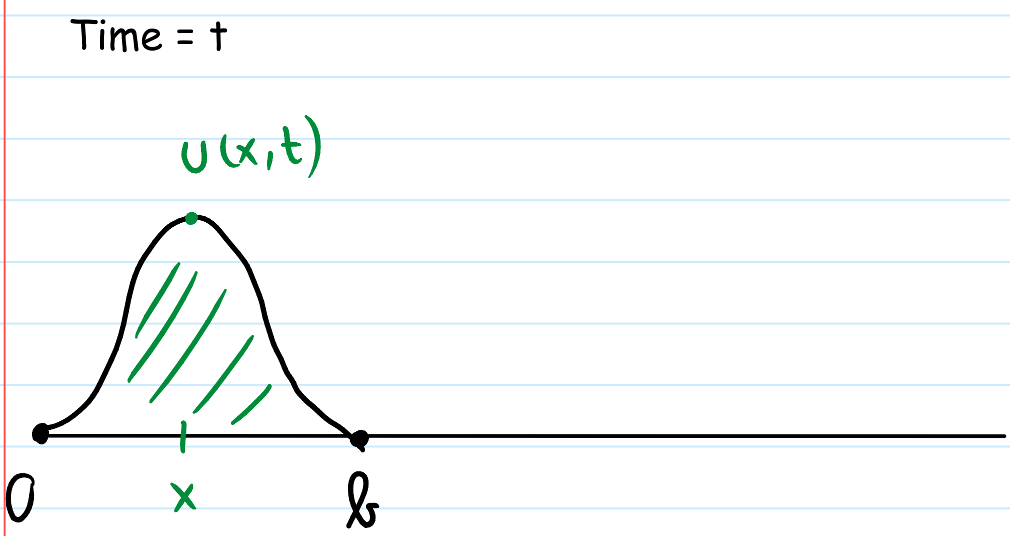


I-DERIVATION (section 1.3)

Let $u(x,t)$ be the density of cars during rush hour, in cars/km (or think of the density of a fluid, in grams per cm)

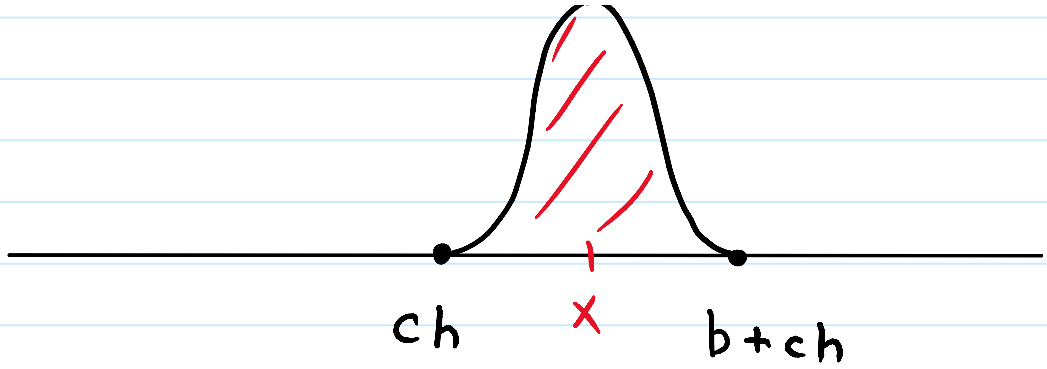
The number of cars (mass of the fluid) on an interval [0,b] at time t is defined to be:

$$\boxed{M = \int_0^b u(x,t)\, dx}$$



In our model, the cars (fluid) move to the right with speed c.

So, at a later time t + h, the cars shifted from [0,b] to [ch,b+ch]



This time, the number of cars in [ch,b+ch] is:

$$\boxed{M = \int_{ch}^{b+ch} u(x, t+h)\, dx}$$

Now, of course, the number of cars (mass) is conserved, so we should have:

$$M = \int_{0}^{b} u(x,t)\, dx = \int_{ch}^{b+ch} u(x, t+h)\, dx$$

Differentiate this with respect to b (using the fundamental theorem of Calculus)

$$u(b,t) = u(b+ch, t+h)$$

Differentiate this with respect to h:

$$0 = \frac{\partial u}{\partial x} \frac{\partial (b+ch)}{\partial h} + \frac{\partial u}{\partial t} \frac{\partial (t+h)}{\partial h}$$

$$0 = u_x (c) + u_t$$

$$\Rightarrow \boxed{u_t + c u_x = 0}$$

II-SOLUTION

How to solve $u_t + c\, u_x = 0 \Rightarrow c\, u_x + u_t = 0$

The good news is that we've already done all the hard work, because this is precisely the same type of PDE we've been studying so far!

Recall: $a u_x + b u_y = 0 \Rightarrow u(x,y) = f(ay - bx) = f(bx - ay)$ (for a different f)

Here: $c\, u_x + 1\, u_t = 0 \Rightarrow u(x,t) = f(1x - ct)$

$u(x,t) = f(x - ct)$

Question: What is f?

Notice:

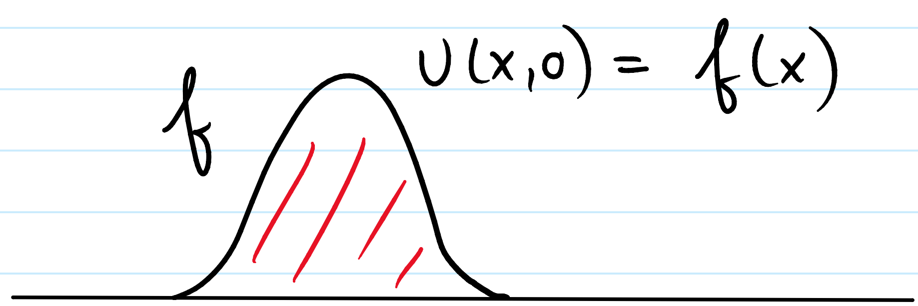
$u(x,0) = f(x-c0) = f(x)$

So $f(x) = u(x,0)$ is the initial profile/density (that is, the density/number of cars at time 0)

Note: (Math 2A) $f(x-a)$ is just $f(x)$ shifted to the right by a units

So $u(x,t) = f(x-ct)$ tells us that with time, the initial profile moves to the right with speed c, just like we wanted!

Time $t = 0$



Time t

This concludes our discussion of first order PDEs (although this is really just the tip of the ice berg).Now let's go back and talk about more general facts about PDEs.

III- INITIAL AND BOUNDARY CONDITIONS (section 1.4)

Recall: (Math 3D) ODEs are usually equipped with initial conditions:

Ex: $y'' + y = 0$ with $y(0) = 3$ and $y'(0) = 4$

The same goes with a PDE, it is usually equipped with one or more initial conditions (IC)

Ex 1: $u_t + c\, u_x = 0$ with $u(x,0) = f(x)$ (initial position/profile)

Ex 2: $u_t + c\, u_x = 0$ with $u_t(x,0) = g(x)$ (initial velocity)

Ex 3: $u_{tt} = u_{xx}$ (Wave equation: Models propagation of waves) with:

$u(x,0) = g(x)$ (initial position)
$u_t(x,0) = h(x)$ (initial velocity) (IC)

(More on that in section 2.1)

Note: **Second**-order PDEs usually require **two** initial conditions

Because PDEs depend **both** on time t **and** on position x, we have a

NEW FEATURE: BOUNDARY conditions (BC), namely position/velocity at **endpoints**!

Ex: $u_t = u_{xx}$

$u = u(x,t)$

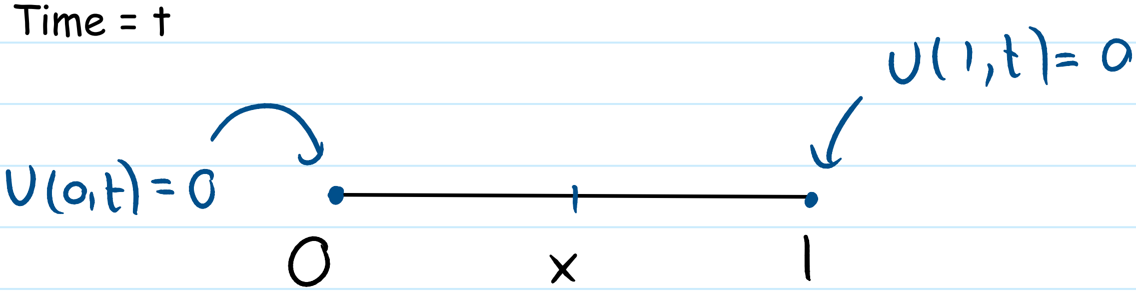
With $t \geq 0$ and $0 \leq x \leq 1$

But this time, we have:

$$u(0,t) = 0 \text{ and } u(1,t) = 0 \quad \text{(BC)}$$

Physical interpretation: u(x,t) is the temperature of a metal rod of length 1, and (BC) says that the rod is insulated in such a way that the temperature at the endpoints is 0.

Time = t



Types of Boundary Conditions (1 dimension)

1) **Dirichlet BC**: You specify values of u at endpoints

Ex: Same but $u(0,t) = t^2$ and $u(1,t) = e^t$ (D)

(temperature at endpoints gets progressively hotter over time)

2) **Neumann BC**: You specify values of u_x (= velocity/rate of change) at endpoints

Ex: $u_x(0,t) = 1$ and $u_x(1,t) = 2$ (N)

3) **Robin BC**: You specify values of $u_x + c\, u$ at boundary

Ex: $u_x(0,t) + 2\, u(0,t) = 0$ and $u_x(1,t) - 3\, u(1,t) = 0$ (R)

Note: We will never talk about Robin BC ever again!

Note: Can mix the BC up, and even mix BC and IC

What about higher dimensions?

For this, we'll need a quick detour into the wonderful world of normal vectors