Today: I would like to discuss an important example of a first-order linear PDE that has a physical significance: The Transport Equation

**Transport Equation:** $u = u(x, t)$, $c = \text{constant}$

\[ u_t + c u_x = 0 \]

Intuitively: Represents density of fluid when it is transported at a speed $c$

**I-DERIVATION** (section 1.3)

Let $u(x,t)$ be the density of cars during rush hour, in cars/km (or think of the density of a fluid, in grams per cm)
The number of cars (mass of the fluid) on an interval \([0,b]\) at time \(t\) is defined to be:

\[
M = \int_{0}^{b} u(x,t) \, dx
\]

Time = \(t\)

In our model, the cars (fluid) move to the right with speed \(c\).

So, at a later time \(t + h\), the cars shifted from \([0,b]\) to \([ch, b+ch]\)

Time \(t + h\)
This time, the number of cars in \([ch,b+ch]\) is:

\[
M = \int_{ch}^{b+ch} U(x, t+h) \, dx
\]

Now, of course, the number of cars (mass) is conserved, so we should have:

\[
M = \int_{0}^{b} U(x, t) \, dx = \int_{ch}^{b+ch} U(x, t+h) \, dx
\]

Differentiate this with respect to \(b\) (using the fundamental theorem of Calculus)

\[
U(b, t) = U(b+ch, t+h)
\]
Differentiate this with respect to $h$:

$$0 = \frac{dU}{dx} \frac{d(b+ch)}{dh} + \frac{dU}{dt} \frac{d(t+h)}{dh}$$

$$0 = U_x (c) + U_t$$

$$\Rightarrow \quad U_t + c U_x = 0$$

**II-SOLUTION**

How to solve $u_t + c u_x = 0$ implies $c u_x + u_t = 0$

The good news is that we've already done all the hard work, because this is precisely the same type of PDE we've been studying so far!

Recall: $au_x + bu_y = 0 \Rightarrow u(x,y) = f(ay - bx) = f(bx - ay)$ (for a different $f$)

Here: $c u_x + 1 u_t = 0 \Rightarrow u(x,t) = f(1x - ct)$

$$u(x,t) = f(x - ct)$$
**Question:** What is \( f \)?

**Notice:**

\[
u(x,0) = f(x - c0) = f(x)
\]

So \( f(x) = u(x,0) \) is the initial profile/density (that is, the density/number of cars at time 0)

**Note:** (Math 2A) \( f(x-a) \) is just \( f(x) \) shifted to the right by \( a \) units

So \( u(x,t) = f(x-ct) \) tells us that with time, the initial profile moves to the right with speed \( c \), just like we wanted!

**Time \( t = 0 \)**

\[
\begin{align*}
u(x,0) &= f(x) \\
\end{align*}
\]

**Time \( t \)**

\[
\begin{align*}
u(x,t) &= f(x-c) \\
\end{align*}
\]
This concludes our discussion of first order PDEs (although this is really just the tip of the iceberg). Now let’s go back and talk about more general facts about PDEs.

### III - Initial and Boundary Conditions (section 1.4)

**Recall:** (Math 3D) ODEs are usually equipped with initial conditions:

*Ex:* \( y'' + y = 0 \) with \( y(0) = 3 \) and \( y'(0) = 4 \)

The same goes with a PDE, it is usually equipped with one or more initial conditions (IC)

*Ex 1:* \( u_t + c \ u_x = 0 \) with \( u(x,0) = f(x) \) (initial position/profile)

*Ex 2:* \( u_t + c \ u_x = 0 \) with \( u_t(x,0) = g(x) \) (initial velocity)

*Ex 3:* \( u_{tt} = u_{xx} \) (*Wave equation: Models propagation of waves*) with:

\[
\begin{aligned}
  u(x,0) &= g(x) \text{ (initial position)} \\
  u_t(x,0) &= h(x) \text{ (initial velocity)} \\
\end{aligned}
\]

(More on that in section 2.1)

**Note:** Second-order PDEs usually require two initial conditions
Because PDEs depend **both** on time \( t \) and on position \( x \), we have a

**NEW FEATURE:** **BOUNDARY** conditions (BC), namely position/velocity at **endpoints**!

**Ex:** \( u_t = u_{xx} \)

\[
\begin{align*}
  u &= u(x,t) \\
  \text{With } t &> 0 \text{ and } 0 \leq x \leq 1 \\
  \text{But this time, we have:} \\
  \quad u(0,t) = 0 \text{ and } u(1,t) = 0 \quad (\text{BC})
\end{align*}
\]

**Physical interpretation:** \( u(x,t) \) is the temperature of a metal rod of length 1, and (BC) says that the rod is insulated in such a way that the temperature at the endpoints is 0.

**Types of Boundary Conditions** (1 dimension)

1) **Dirichlet BC:** You specify values of \( u \) at endpoints
Ex: Same but \( u(0,t) = t^2 \) and \( u(1,t) = e^t \) \( \text{(D)} \)

(temperature at endpoints gets progressively hotter over time)

2) **Neumann BC**: You specify values of \( u_x \) (= velocity/rate of change) at endpoints

Ex: \( u_x(0,t) = 1 \) and \( u_x(1,t) = 2 \) \( \text{(N)} \)

3) **Robin BC**: You specify values of \( u_x + c u \) at boundary

Ex: \( u_x(0,t) + 2u(0,t) = 0 \) and \( u_x(1,t) - 3u(1,t) = 0 \) \( \text{(R)} \)

**Note**: We will never talk about Robin BC ever again!

**Note**: Can mix the BC up, and even mix BC and IC

What about higher dimensions?

For this, we'll need a quick detour into the wonderful world of normal vectors