LECTURE 5: LINEAR INDEPENDENCE

Monday, October 7, 2019 11:38 AM

I- LINEAR DEPENDENCE

Today's topic du jour is linear independence, which is connected to $Ax = 0$ in an interesting way.

Example: Are the following vectors related?

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ ?}$$

<u>YES</u> : $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Rewriting this in terms of the 0 vector, this gives:

$$\underline{1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$au + bv + cw = \mathbf{0}$$

Definition: {u,v,w} is linearly dependent (LD) if there are numbers a, b, c, <u>not all 0</u>, such that:

$$\boxed{au + bv + cw = \mathbf{0}}$$

In other words, there is some linear combo that gives you the **0** vector.

In the above example, a = 1, b = -2, c = -3, so linearly dependent

Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

$$0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a = 0, b = 2, c = -1, so LD

Note:

1) It's ok if *some* of the coefficients are 0, just not *all* of them.
2) Notice that the first vector is not a linear combo of the other vectors!

Example: Is {u,v,**0**} linearly dependent?

Yes: $0\,u + 0\,v + 1\,\mathbf{0} = \mathbf{0}$

($a = 0$, $b = 0$, $c = 1$, not all 0)

Fact: In fact, anything with **0** is LD

Question: Why do we require a, b, c to be not all 0?

Because you can always write $\underbrace{0\,u + 0\,v + 0\,w}_{\text{"Trivial" linear combo}} = \mathbf{0}$

"Trivial" linear combo
($a = 0$, $b = 0$, $c = 0$)

Which motivates the definition of linear independence:

II- LINEAR INDEPENDENCE

Definition: {u,v,w} is linearly independent (LI) if:

$$au + bv + cw = \mathbf{0} \Rightarrow a = 0,\ b = 0,\ c = 0$$

That is, the **only** way of getting the **0** vector is with the trivial linear combo.

Example: Is the following set LI?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Suppose

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then:

$$\begin{cases} a + b + c = 0 \\ \phantom{a+{}} b + c = 0 \\ \phantom{a+b+{}} c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

MAGIC

=> LI

III- Ax = 0

It turns out there's a really cool connection between LI and Ax = 0

Example: LD or LI ?

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \\ 7 \end{bmatrix} \right\}$$

Are there a, b, c (not all 0) such that:

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} -1 \\ 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ ?$$

SAME AS:

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 4 \\ 0 & 1 & 3 \\ 1 & 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$$\Rightarrow \quad A\underline{x} = \underline{0} \quad !!!$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 8 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

STUPID WAY:

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back sub:

$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \Rightarrow \underline{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=> LI

SMART WAY:

In REF, there's a pivot in every column of A
=> No free variables
=> Ax = 0 has only one solution: x = 0
=> LI

And this is such an important fact, we'll isolate it in a theorem:

IV- THE COLUMN THEOREM

THE COLUMN THEOREM:

1) A has a pivot in every column
$\Leftrightarrow$ 2) Ax = 0 => x = 0
$\Leftrightarrow$ 3) The columns of A are LI

$\Leftrightarrow$ 2) $Ax = 0 \Rightarrow x = 0$

$\Leftrightarrow$ 3) The columns of A are LI

COLUMNS

$Ax = 0$

LI

ROWS

$Ax = b$

SPAN

Example: For which h is the following LD ?

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 38+h \end{bmatrix}$$

If $38 + h$ is not 0, then have 3 pivots

=> No free var

=> LI

So for LD, want $38 + h = 0$

Answer: $h = -38$

Example: Is the following set LD?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

Fact: 3 vectors in R^2 are automatically LD $(3 > 2)$

Why? A has at most 2 pivots, so at least 1 free variable

$$\begin{bmatrix} 1 & 4 & 1 \\ 1 & 8 & 3 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

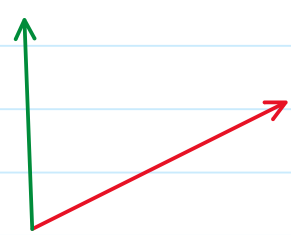
V- LI AND SPAN

Note: LI vectors are nice because their span is exactly what you think it is:

Example 1:

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\} \quad \text{is a } \textbf{plane} \text{ in } R^3$$

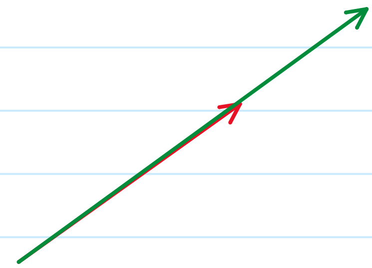
LI



Example 2:

$$\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$$ is not a plane, but a **line**

LD



LD sets are redundant; you can always remove LD vectors without changing the span (like Jenga)

Example 2 (again)

$$\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\cancel{\begin{bmatrix}2\\2\\2\end{bmatrix}}\right\} = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

You'd never invest in LD vectors, they give you no new info

But LI sets are essential, removing a LI vector changes the span

(Ex: Try removing a vector in Example 1 and see what you get)

So LI vectors are a good investment, all their info is essential

(Section 2.9: Will see exactly which vectors you can remove)