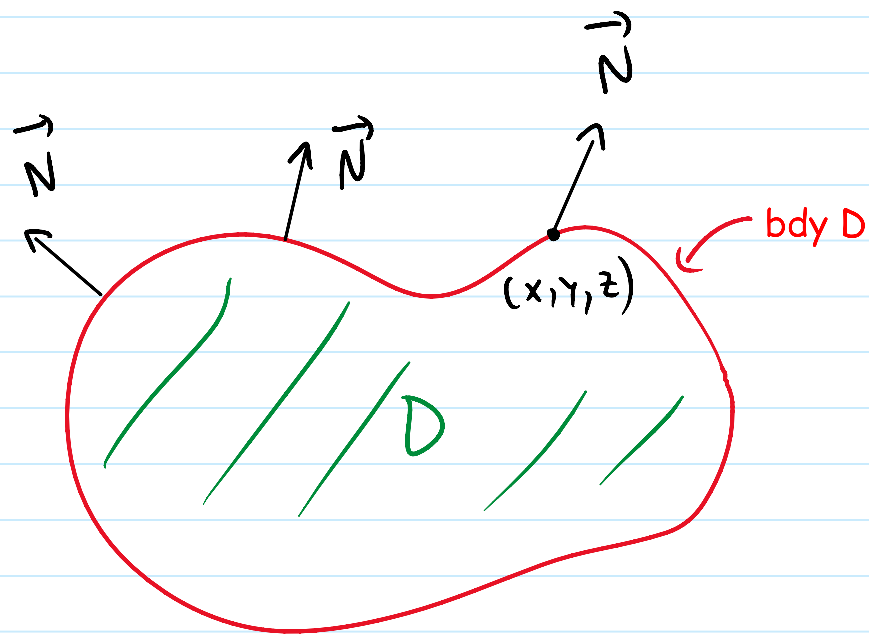
LECTURE 5: MORE FUN PDE FACTS

Saturday, October 5, 2019 8:54 PM

I- REVIEW: NORMAL VECTORS

Suppose D is a region in R^2 or R^3 with boundary bdy D

(Think for instance D = ball and bdy D = Sphere)



At each point on bdy D, there is:

n = outside-pointing unit normal vector
(unit = length 1, normal = perpendicular to the surface)

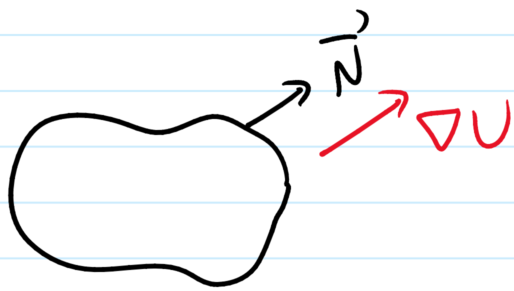
Definition: Normal derivative:

$$\boxed{\frac{\partial u}{\partial N} = \nabla u \cdot \vec{N}}$$

(= Directional derivative in the direction of the normal vector)

Interpretation: $\frac{\partial u}{\partial N}$ measures how much u flows in and out of D

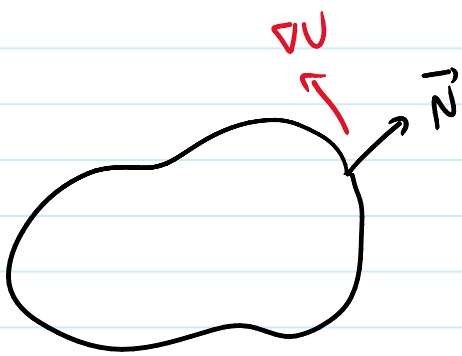
Ex 1:



$$\frac{\partial u}{\partial N} > 0$$

(u flows out of D)

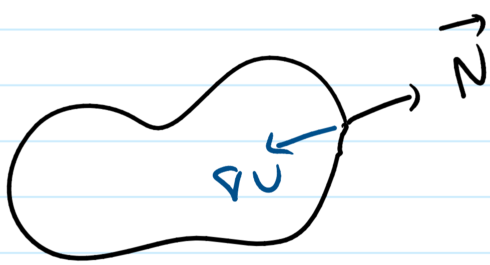
Ex 2:



$$\frac{\partial u}{\partial N} = 0$$

(u is stuck on bdy D, like super glue)

Ex 3:

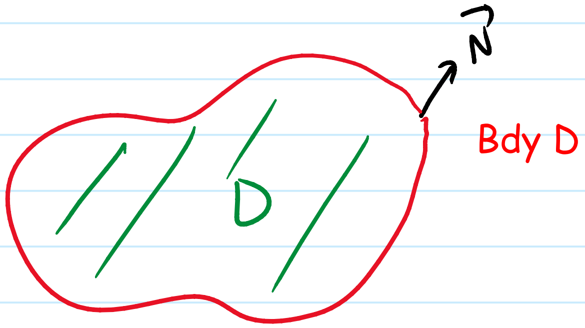


$$\frac{\partial u}{\partial N} < 0$$

(u flows into D)

Note: Divergence Theorem:

$$\boxed{\iint_{\text{Bdy } D} F \cdot \vec{N}\, dS = \iiint_D \mathrm{DIV}(F)\, dx\,dy\,dz}$$



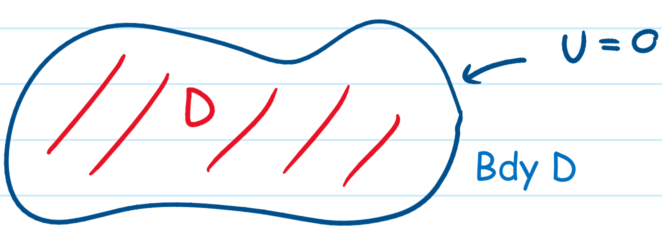
II- BOUNDARY CONDITIONS IN HIGHER DIMENSIONS

What kinds of boundary conditions are there in higher dimensions?

Types of boundary conditions:

1) **Dirichlet**: You specify u on bdy D

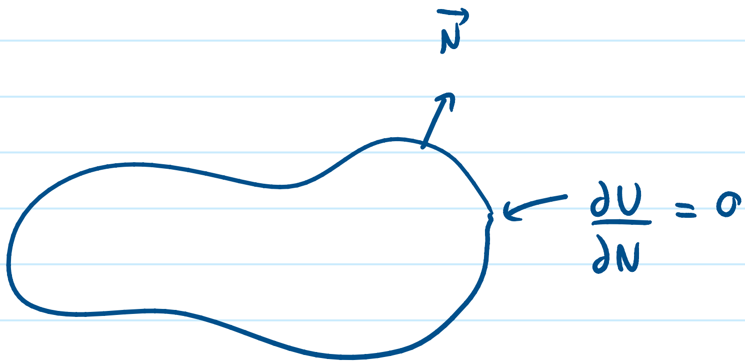
Ex 1: u = 0 on bdy D



(Think insulating a metal sphere to have temperature 0)

2) **Neumann**: You specify $\frac{\partial u}{\partial N}$ on bdy D

Ex 2: $\frac{\partial u}{\partial N}$ = 0 on on bdy D



(Think of making the boundary sticky so that u doesn't move)

3) **Robin**: You specify $u + c\,\frac{\partial u}{\partial N}$ on bdy D

Ex 3: $u - \frac{\partial u}{\partial N} = 0$ on bdy D (u equals to its normal velocity)

There are more exotic BC, see book if you're interested. Most "famous" boundary condition is for Euler's equations, where you control your pressure to have 0 divergence.

We'll mainly focus on the 1d case with just initial conditions or Dirichlet/Neumann boundary conditions.

III- EXISTENCE AND UNIQUENESS (Section 1.5)

Let's continue with more fun generalities about PDEs.

Given that this is an intro to PDE class, you might wonder: What are some of the big questions of PDE? Here are the main ones (and I'll illustrate them with ODE examples):

1) **Existence**: Does a PDE have a solution or not? Sometimes it might not!

Example: $y'' + y = 0$ with $y(0) = 0$ and $y(\pi) = 1$

$y(t) = A\cos(t) + B\sin(t)$ (Math 3D)

$y(0) = A(1) + B(0) = A = 0$ (by $y(0) = 0$)

So $y(t) = B\sin(t)$

But then $y(\pi) = B\sin(\pi) = 0$, so how can $y(\pi) = 1$??? (contradiction)

So this simple ODE (with boundary conditions) has no solution, and the same thing can happen with PDEs.

2) **Uniqueness**: Could a PDE have many solutions? Absolutely!

Example: $y'' + y = 0$ but $y(0) = 0$ and $y(\pi) = 0$

In this case we still have $y(t) = B\sin(t)$

But then automatically $y(\pi) = B\sin(\pi) = 0$, so this ODE has **INFINTELY** many solutions, namely $y(t) = B\sin(t)$ for any B

3) **Sensitive dependence to initial/boundary conditions**: If we change the initial/boundary conditions just a little bit, does the solution also change just a little bit?

Example: $y'' - y = 0$ but $y(0) = 1$ and $y'(0) = -1$ for $n = 1, 2, \ldots$

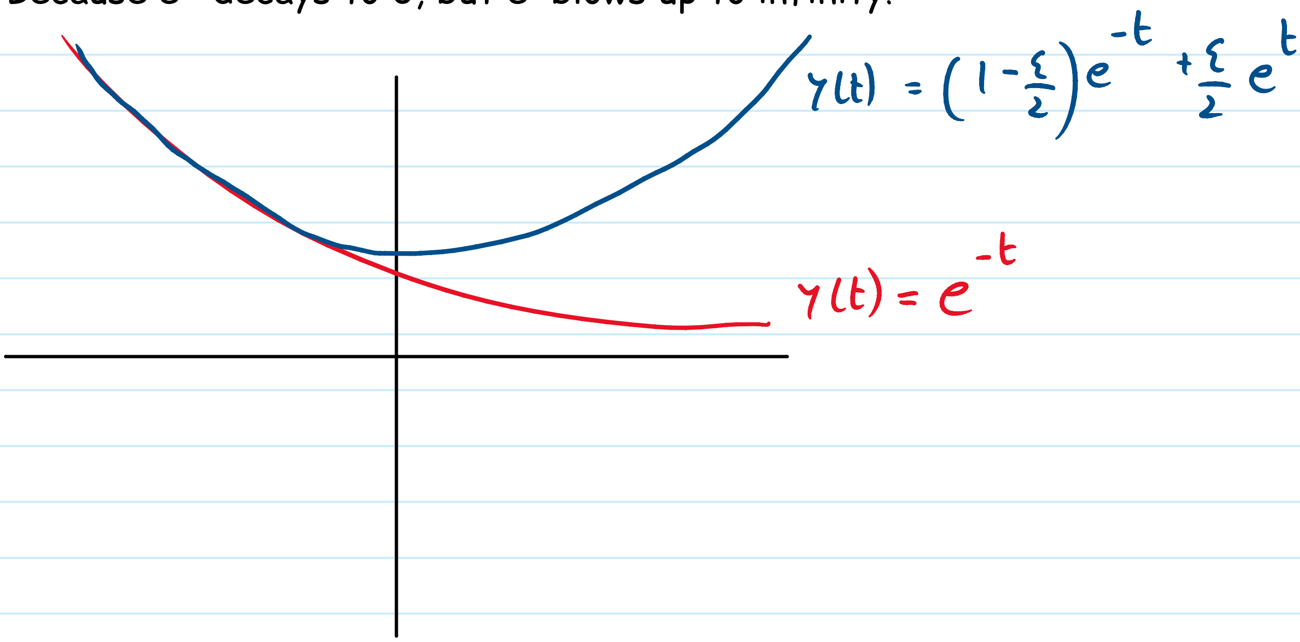
Can show: $y(t) = e^{-t}$

But now, what if we change the initial conditions to $y(0) = 1$ and $y'(0) = -1 + \varepsilon$?

Then the solutions are

$$y(t) = \left(1 - \frac{\varepsilon}{2}\right) e^{-t} + \frac{\varepsilon}{2} e^{t}$$

Which is a big problem if you think about it...
Because e^{-t} decays to 0, but e^{t} blows up to infinity!



So the perturbed solution isn't close to the original solution at all! (for large t)

You might say: Big deal, why would we change $y'(0)$ from -1 to $-1 + \varepsilon$ anyway? Well, think about it! When you make a physical measurement, or when you input -1 in a computer, you never really get exactly -1, but instead -1.0000001 or so. And this implies here that the solution you see isn't at all the solution you're supposed to have!

IV- TYPES OF SECOND-ORDER PDE (section 1.6)

So far: Solved first-order PDEs, and third-order ones are too complicated anyway, so for the rest of the course (and probably 112B & C), we'll focus on second order PDEs. Turns out there's a nice classification of them:

Suppose you have a PDE of the form:

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + \underbrace{?\,u_x + ?\,u_y + ?\,u = f}_{\text{"JUNK"}}$$

Definition:

1) If $\mathcal{D} = b^2 - 4ac < 0$, then the PDE is elliptic
2) If $\mathcal{D} = b^2 - 4ac > 0$, then the PDE is hyperbolic
3) If $\mathcal{D} = b^2 - 4ac = 0$, then the PDE is parabolic

Mnemonic:

Ellipse (Circle): $x^2 + y^2 = 1$ which means $1\,x^2 + 0\,xy + 1\,y^2 = 1$

And can check $\mathcal{D} = 0^2 - 4(1)(1) = -4 < 0$

Example: What is the type of the PDE

$$5u_{xx} + 6u_{xy} + 4u_{yy} + 3u_x + 5u = x^2$$

$\mathcal{D} = 6^2 - 4(5)(4) = 36 - 80 = -44 < 0$, so elliptic

Most famous PDE and their types:

1) Laplace's equation: $u_{xx} + u_{yy} = 0$

$\mathcal{D} = 0^2 - 4(1)(1) = -4 < 0$, so elliptic

2) Wave equation: $u_{tt} = u_{xx} \Rightarrow u_{xx} - u_{tt} = 0$

$\mathcal{D} = 0^2 - 4(1)(-1) = 4 > 0$, so hyperbolic

3) Heat equation: $u_t = u_{xx} \Rightarrow u_{xx} + 0\, u_{tt} + 1\, u_t$

$\mathcal{D} = 0^2 - 4(1)(0) = 0$, so parabolic

Fun Fact: With a change of coordinates, can turn any elliptic PDE ($D < 0$) into $u_{xx} + u_{yy} + \text{JUNK} = 0$

Why? Use the coordinate method with the coordinates

$$\begin{cases} \xi = x \\ \eta = \dfrac{y - b/2}{\sqrt{-D}} \end{cases}$$

(See book if you're curious about the details and about why you choose that change of coordinates)

OPTIONAL: The *REAL* reason:

$$a\,U_{xx} + b\,U_{xy} + c\,U_{yy} + \text{JUNK} = f$$

Can put the second order coefficients in a matrix:

$$A = \begin{array}{cc} u_{xx} & u_{xy} \\ \multicolumn{2}{c}{\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}} \\ u_{yx} & u_{yy} \end{array}$$

(Nice, because symmetric)

Example: $u_{xx} + u_{yy} = 0 \Rightarrow 1\, u_{xx} + 0 u_{xy} + 1\, u_{yy} = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: 1, 1 (all positive)

FACT:

1) If all the eigenvalues of A are positive, then the PDE is elliptic
2) If A has a positive and a negative eigenvalue, then the PDE is hyperbolic
3) If A has a zero eigenvalue, then the PDE is parabolic

(Compare to Math 121B with quadratic forms: an ellipse also corresponds to positive eigenvalues, etc.)