# LECTURE 6: THE WAVE EQUATION

Monday, October 7, 2019 10:03 PM

## I- TYPES OF SECOND-ORDER PDE (Section 1.6)

Recall: PDE of the form

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + \text{JUNK} = f$$

**Definition**:

1) If $\mathscr{D} = b^2 - 4ac < 0$, then the PDE is elliptic
2) If $\mathscr{D} = b^2 - 4ac > 0$, then the PDE is hyperbolic
3) If $\mathscr{D} = b^2 - 4ac = 0$, then the PDE is parabolic

**Fact**: With a change of coordinates, can turn any elliptic PDE ($\mathscr{D} < 0$) into $u_{xx} + u_{yy} + \text{JUNK} = 0$ (No more $u_{xy}$ term)

**Why**? Use the coordinate method with the coordinates

$$\begin{cases} x' = x \\ y' = \dfrac{y - b/2}{\sqrt{-\mathscr{D}}} \end{cases}$$

(See book if you're curious about the details and about why you choose that change of coordinates)

So any elliptic PDE is more or less like Laplace's equation!

Which is why we generally just study Laplace's equation instead of all elliptic PDE

**Note: The *REAL* reason:**

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + \text{JUNK} = f$$

Put the second order coefficients in a matrix:

$$A = \begin{array}{cc} u_{xx} & u_{xy} \\ \multicolumn{2}{c}{\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}} \\ u_{yx} & u_{yy} \end{array}$$

(Nice, because symmetric)

**Example**: $u_{xx} + u_{yy} = 0 \Rightarrow 1\,u_{xx} + 0u_{xy} + 1\,u_{yy} = 0$ ($a = 1$, $b = 0$, $c = 1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: 1, 1 (all positive)

**FACT:**

1) If all the eigenvalues of $A$ are positive, then the PDE is elliptic
2) If $A$ has a positive and a negative eigenvalue, then the PDE is hyperbolic

3) If A has a zero eigenvalue, then the PDE is parabolic

(Compare to quadratic forms in Math 121B: an ellipse also corresponds to positive eigenvalues)

And this officially concludes Chapter 1, which was an intro to PDEs and a study of first-order PDEs. For the rest of the course, we will study the three most famous PDE, starting with the wave equation!

## II- THE WAVE EQUATION (Section 2.1)

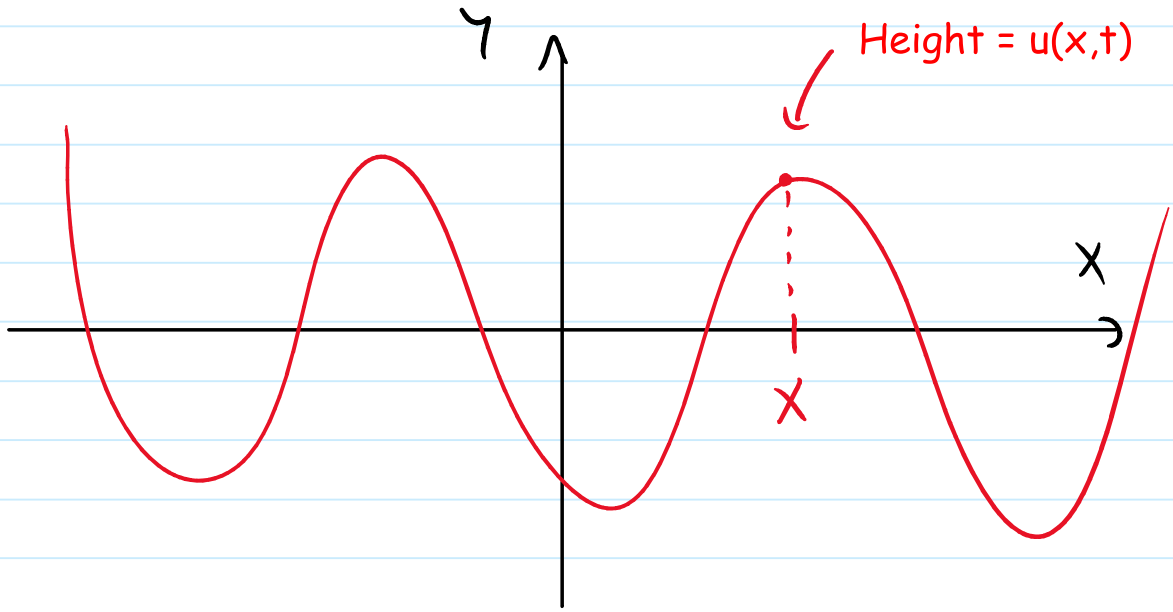
Studies behavior of a wave, which could be a vibrating string, sound, light wave, water wave/ripple, vibrating drum (2 dimensions), vibrating solid (3 dimensions)

**Wave Equation:** $u = u(x,t)$

$$u_{tt} = c^2 u_{xx}$$

**Interpretation:** $u$ is the height of a wave at position $x$ and time $t$

**Picture**: At time = $t$




We will derive it at some point, but first let's solve it!

## III- THE FACTORING METHOD

It is all based on the following clever idea:

$t^2 - c^2 x^2 = (t-cx)(t+cx)$

And we will apply this idea, but with operators:

Definition: $$\frac{\partial}{\partial t} u = u_t$$

"Operator" which takes a function u as its input and returns $u_t$

In particular:

$$\left(\frac{\partial}{\partial t}\right)^2 u = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} u\right) = \frac{\partial}{\partial t} u_t = u_{tt}$$

(Compare to $T^2(u)$ in Math 3A or 121A)

STEP 1:

In particular, we get:

$$u_{tt} - c^2 u_{xx} = \left(\left(\frac{\partial}{\partial t}\right)^2 - c^2\left(\frac{\partial}{\partial x}\right)^2\right) u$$

$$= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

**CHECK:**

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

$$= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(U_t + cU_x)$$

$$= \frac{\partial}{\partial t}(U_t + cU_x) - c\frac{\partial}{\partial x}(U_t + cU_x)$$

$$= U_{tt} + \cancel{cU_{xt}} - \cancel{cU_{tx}} - c^2 U_{xx}$$

$$= U_{tt} - c^2 U_{xx}$$

STEP 2: Therefore $u_{tt} - c^2 u_{xx} = 0$ simply becomes:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left[\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)U\right]}_{V} = 0$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)V = 0$$

$$\Rightarrow \quad \boxed{v_t - c v_x = 0}$$

WHOA!

So essentially we just turned the wave equation into the transport equation!

STEP 3: Solve $v_t - c\, v_x = 0$

From the lecture on the transport equation, we get:

$v(x,t) = f(x-(-c)t) = f(x+ct)$

STEP 4: Finally, use the definition of v:

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

$$= u_t + c\, u_x$$

$$\Rightarrow \quad u_t + c\, u_x = v$$

$$\Rightarrow \quad \boxed{u_t + c\, u_x = f(x+ct)}$$

## ANOTHER TRANSPORT EQUATION!!!

STEP 5: This time it's an inhomogeneous transport equation!

**Recall**: $u = u_0 + u_p$ where:

$u_0$ solves $u_t + c\, u_x = 0 \Rightarrow u_0(x,t) = G(x-ct)$ ($G$ = arbitrary)

$u_p$ is *one* solution of $u_t + c\, u_x = f(x+ct)$

**Guess**: $u_p = h(x+ct)$ for some function $h$

Check: Plug into $u_t + c\, u_x = f(x+ct)$

$$\left(h(x+ct)\right)_t + c\left(h(x+ct)\right)_x = f(x+ct)$$

$$h'(x+ct)(c) + c\, h'(x+ct) = f(x+ct)$$

$$2c\, h'(x+ct) = f(x+ct)$$

$$\Rightarrow \quad 2c\, h' = f$$

$$\Rightarrow \quad h' = \frac{1}{2c} f$$

$$\Rightarrow \quad h = \frac{1}{2c} F$$ ($F$ = antiderivative of $f$)

$$\Rightarrow u_p(x,t) = \frac{1}{2c} F(x+ct)$$

STEP 6: General solution:

$$u(x,t) = G(x-ct) + (1/2c)\,F(x+ct)$$
$$= G(x-ct) + F(x+ct) \text{ (for some other arbitrary F)}$$

CONCLUSION:

The general solution of $u_{tt} = c^2 u_{xx}$ is:

$$u(x,t) = F(x+ct) + G(x-ct)$$

(where F and G are arbitrary)

**Physical Interpretation:**

A wave is the sum of two functions, one moving to the left at speed c and the other one moving to the right at speed c