LECTURE 6: LINEAR TRANSFORMATIONS (I)

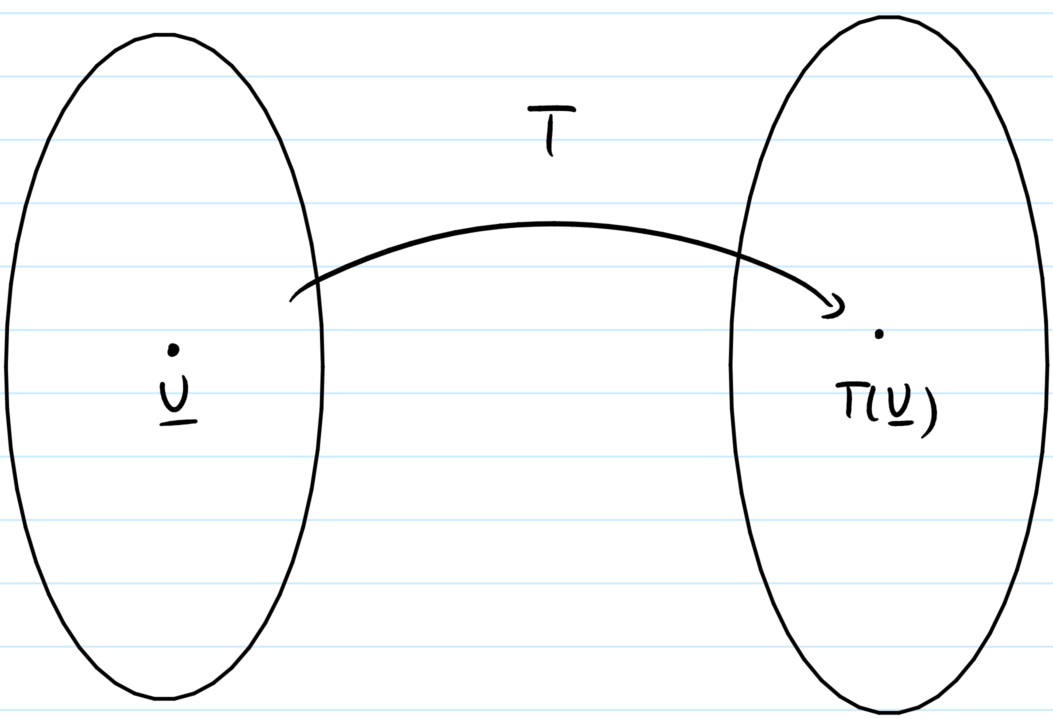
Monday, October 7, 2019 8:53 PM

So far in our Linear Algebra (LA) adventure, we have discussed the first big idea of LA: systems of equations. Today, we'll turn to the second big idea in LA: Linear Transformations

I- DEFINITION AND EXAMPLE (Section 1.8)

Definition: A linear transformation T is a function such that:

1) $T(u+v) = T(u) + T(v)$
2) $T(cu) = c\,T(u)$ (where c is a number)



So a linear transformation is just a function with two special properties.

Example: Show that T is a linear transformation:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

Note: The reason for this strange notation will make sense soon

<u>EX</u> $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 + 2(2) \\ 1 - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

1) LET $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\underline{v} = \begin{bmatrix} z \\ t \end{bmatrix}$

THEN: $T(\underline{u} + \underline{v})$

$$= T\left(\begin{bmatrix} x + z \\ y + t \end{bmatrix}\right)$$

DEF OF T !!!

$$= \begin{bmatrix} (x + z) + 2(y + t) \\ (x + z) - (y + t) \end{bmatrix}$$

$$= \begin{bmatrix} x+z+2y+2t \\ x+z-y-t \end{bmatrix}$$

$$= \begin{bmatrix} x+2y \\ x-y \end{bmatrix} + \begin{bmatrix} z+2t \\ z-t \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} z \\ t \end{bmatrix}\right)$$

$$= T(\underline{u}) + T(\underline{v}) \checkmark$$

2) $$T(c\underline{u}) = T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

DEF OF T

$$= T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right)$$

$$= \begin{bmatrix} cx+2cy \\ cx-cy \end{bmatrix}$$

$$= \begin{bmatrix} c(x+2y) \\ c(x-y) \end{bmatrix}$$

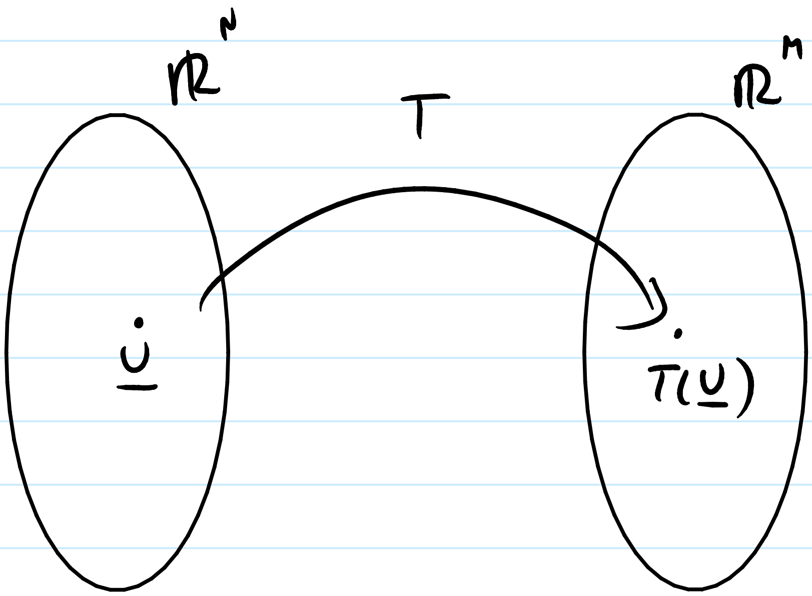
$$= c \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

$$= c\, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= c\, T(\underline{u}) \checkmark$$

Note: Here T: R^2 -> R^2
(Meaning: Input u is in R^2 and output T(u) is in R^2)

In general, T: R^n –> R^m



Mnemonic: INPUT, MOUTHPUT

II- NONLINEAR TRANSFORMATIONS (Section 1.8)

Example: Show that the following transformation is **not** linear

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

⚠ To show something is false, you have to give a specific example that shows it's false

(NOT enough to simply say $x^2 + y^2$ is not $(x+y)^2$

Example:

$$T\left(2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \overset{\text{DEF}}{=} \begin{bmatrix} 2^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

<u>BUT</u>

$$2\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

So T(cu) is not c T(u) => T not linear

Useful Test: If T is linear, then T(**0**) = **0**

(Why? T(**0**) = T(0 **0**) = 0 T(**0**) = **0**)

Example: Show that the following transformation is not linear

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1+x \\ x-y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Finally, it turns out there's a neat way of constructing linear transformations.

III- THE MATRIX OF A LINEAR TRANSFORMATION

(Section 1.9)

Fact: If A is a matrix, then $T(x) = Ax$ is linear.

Ex $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x + 2y \\ 3x + 4y \\ 5x + 6y \end{bmatrix}$$

So *given* a matrix A, we can construct a linear transformation $T(x) = Ax$

MIRACLE: **ANY** linear transformation is of the form $T(x) = Ax$ for some A

(=> COMPLETE classification of LT)

Example: Find the matrix A of T, where:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

Guess: $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

STEP 1: Calculate:

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2(1) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

STEP 2: Put everything together:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

And therefore T(x) = Ax

⚠ IMPORTANT TO DO $T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

AND $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ IN THAT ORDER

Aside: Why this works:

$$T(\underline{x}) = T\begin{bmatrix} x \\ y \end{bmatrix} = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

LINEAR

$$= x\,T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\,T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= A\,\underline{x}$$

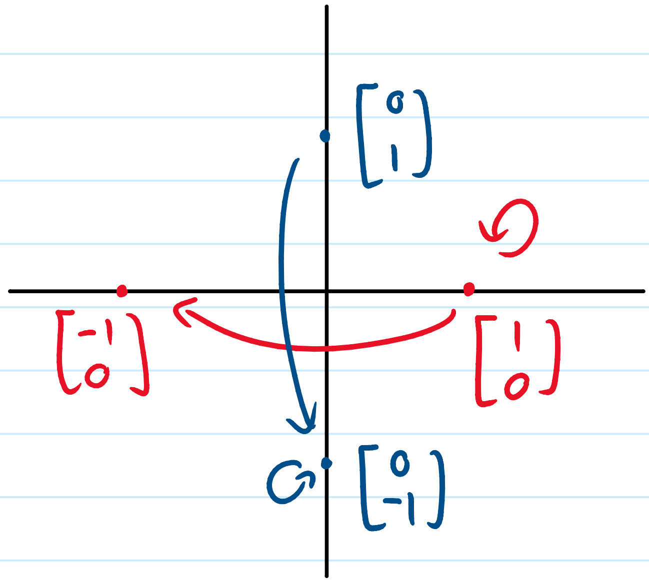
So $T(x) = Ax$

This is AMAZING, btw, because before matrices were just tables of numbers, but now they are alive, because they're linear transformations!!! (Like the Frankenstein of math!)

IV- LINEAR GEOMETRY ? (Section 1.9)

Example 1:

(a) Find the matrix A of T which first reflects points in R^2 about the x axis, and then about the y axis



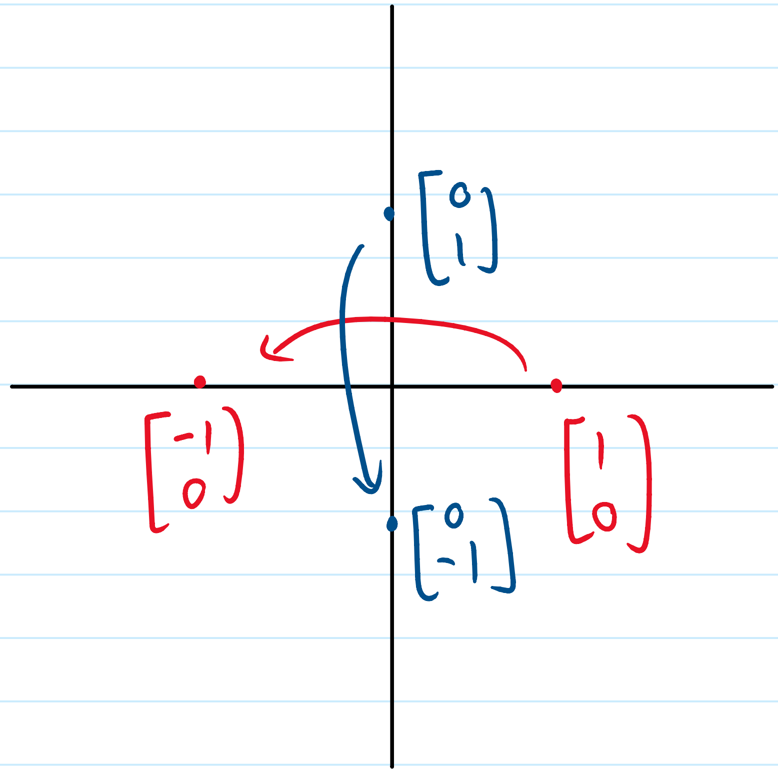
$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) Find a formula for T

$$T\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Example 2: Find the matrix B of S which rotates points in R^2 by 180 degrees (counterclockwise)



$$S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}, \quad S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

$$B = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

Notice: A = B => T = S

In particular, the two geometric transformations are the same!

And **THIS** is the essence of linear algebra: using algebraic techniques (here: matrices) to study geometric things (here: reflections/rotations)