LECTURE 7: THE WAVE EQUATION (II)

Wednesday, October 9, 2019 7:37 PM

I- THE COORDINATE METHOD

Just like first-order PDEs, there is a Chen Lu way of solving the wave equation $u_{tt} = c^2 u_{xx}$, which of course requires to know which coordinates we use.

But this time it's not hard to guess it, given that

$$x^2 - c^2 t^2 = (x-ct)(x+ct)$$

STEP 1:

Define:

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

STEP 2:

Then:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= u_\xi (1) + u_\eta (1)$$

$$= u_\xi + u_\eta$$

$$U_{xx} = \frac{\partial U_x}{\partial x} = \frac{\partial U_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U_x}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= (U_\xi + U_\eta)_\xi \,(1) + (U_\xi + U_\eta)_\eta \,(1)$$

$$= U_{\xi\xi} + U_{\eta\xi} + U_{\xi\eta} + U_{\eta\eta}$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Similarly (see HW)

$$U_{tt} = c^2\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right)$$

STEP 3:

$u_{tt} = c^2 u_{xx}$ then becomes:

$$\cancel{c^2}\left(\cancel{U_{\xi\xi}} - 2U_{\xi\eta} + \cancel{U_{\eta\eta}}\right) = \cancel{c^2}\left(\cancel{U_{\xi\xi}} + 2U_{\xi\eta} + \cancel{U_{\eta\eta}}\right)$$

$$\Rightarrow \quad -2U_{\xi\eta} = 2U_{\xi\eta}$$

$$\Rightarrow \quad 4U_{\xi\eta} = 0$$

$$\Rightarrow \quad U_{\xi\eta} = 0$$

STEP 4:

$$\Rightarrow \quad (U_\xi)_\eta = 0$$

$$\Rightarrow \quad U_\xi = f(\xi)$$

$$\Rightarrow \quad U = F(\xi) + G(\eta)$$

(F = Antiderivative of f, G = arbitrary)

STEP 5: Solution:

$$\Rightarrow \quad \boxed{u(x,t) = F(x-ct) + G(x+ct)}$$

F and G arbitrary

Which is the same solution we got using the factoring method

II- D'ALEMBERT'S FORMULA

Now how about some initial conditions?

Solve:

$$\boxed{\begin{cases} U_{tt} = c^2 U_{xx} \\ U(x,0) = \phi(x) \\ U_t(x,0) = \Psi(x) \end{cases}}$$

$\leftarrow$ Initial Position

$\leftarrow$ Initial Velocity

All you need to do is to plug in your initial conditions in your solution!

STEP 1:

$u(x,t) = F(x-ct) + G(x+ct)$

$u(x,0) = F(x-0) + G(x+0) = F(x) + G(x) = \phi(x)$

So $F(x) + G(x) = \phi(x)$

On the other hand:

$u_t(x,t) = F'(x-ct)(-c) + G'(x+ct)(c)$

$$u_t(x,0) = -cF'(x) + cG'(x) = \psi(x)$$

$$\Rightarrow G' - F' = \psi/c$$

$$\Rightarrow \int_0^x G'(s) - F'(s)\,ds = 1/c \int_0^x \psi(s)\,ds$$

$$\Rightarrow G(x) - F(x) - \underbrace{(G(0) - F(0))}_{A} = 1/c \int_0^x \psi(s)\,ds$$

Hence, we get:

$$\begin{cases} G(x) + F(x) = \phi(x) \\ G(x) - F(x) = A + 1/c \displaystyle\int_0^x \psi(s)\,ds \end{cases}$$

STEP 2: Solve this:

Add both equations to get:

$$2G(x) = \phi(x) + A + 1/c \int_0^x \psi(s)\,ds$$

$$\Rightarrow G(x) = 1/2\,\phi(x) + A/2 + 1/(2c) \int_0^x \psi(s)\,ds$$

Subtract both equations to get:

$$2F(x) = \phi(x) - A - 1/c \int_0^x \psi(s)\,ds$$

$$\Rightarrow F(x) = 1/2\,\phi(x) - A/2 + 1/(2c) \int_x^0 \psi(s)\,ds$$

STEP 3:

Therefore, we get:

$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= 1/2\ \phi(x-ct) - \cancel{A/2} + 1/(2c) \int_{x-ct}^{0} \psi(s)\, ds$$
$$+ 1/2\ \phi(x+ct) + \cancel{A/2} + 1/(2c) \int_{0}^{x+ct} \psi(s)\, ds$$

$$= 1/2\ (\phi(x-ct) + \phi(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds$$

STEP 4: CONCLUSION

D'ALEMBERT'S FORMULA:

$$\begin{cases} U_{tt} = c^2 U_{xx} \\ U(x,0) = \phi(x) \\ U_t(x,0) = \psi(x) \end{cases} \implies$$

$$U(x,t) = \frac{1}{2}\Big(\phi(x-ct) + \phi(x+ct)\Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds$$

Interpretation: The initial position splits up into two, one moving to the right and the other one moving to the left, and the contribution of the velocity is over the whole interval [x-ct,x+ct]

III- EXAMPLES

Example 1: $u_{tt} = u_{xx}$ with $u(x,0) = 0$ and $u_t(x,0) = \cos(x)$

D'Alembert gives us:

$$u(x,t) = 1/2\,(0 + 0) + 1/2 \int_{x-t}^{x+t} \cos(s)\,ds$$

$$= 1/2\,[\sin(x+t) - \sin(x-t)]$$

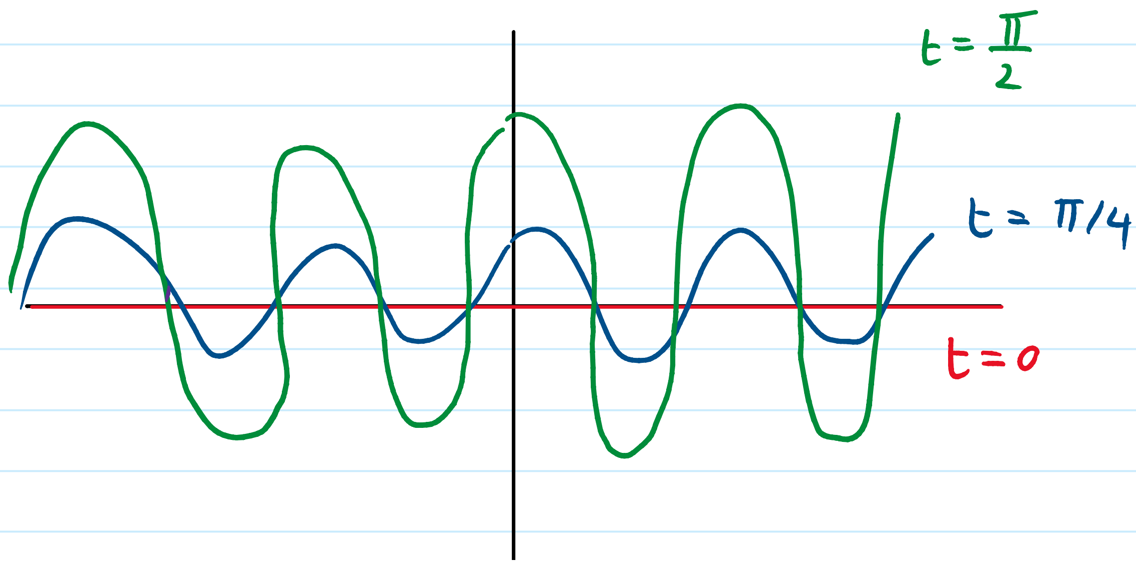
$$= 1/2\,[\sin(x)\cos(t) + \cos(x)\sin(t) - \sin(x)\cos(t) + \cos(x)\sin(t)]$$

$$= 1/2\,[2\cos(x)\sin(t)]$$

$$= \cos(x)\sin(t)$$

$$= \sin(t)\cos(x)$$

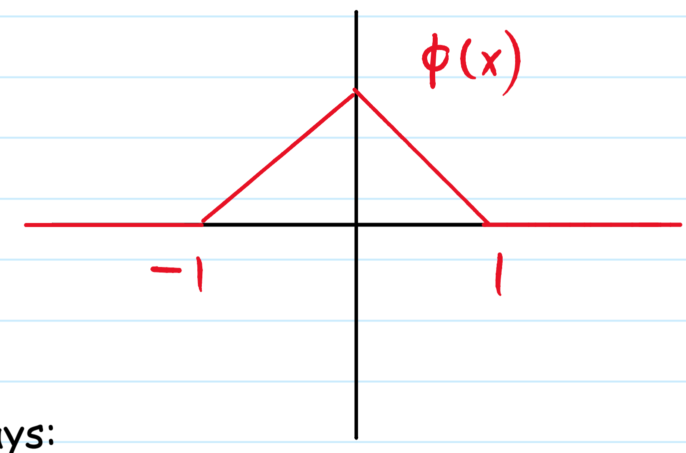
The solution starts at 0 (when $t = 0$), and then looks more and more like cos(x) (when $t = \pi/2$), then goes back to 0 (when $t = \pi$), then looks like -cos(x) (when $t = 3\pi/2$), then goes back to 0, and so on. So it's like an oscillating cos curve!



Example 2: [The plucked string]

$u_{tt} = u_{xx}$ ($c = 1$) with $u(x,0) = \phi(x)$ and $u_t(x,0) = 0$, where:

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ 1 - |x| & \text{for } -1 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$



D'Alembert says:

$u(x,t) = 1/2\ [\ \phi(x-t) + \phi(x+t)\ ]$

Now given the piecewise definition of ϕ, this becomes quite complicated, and we need to split this up into a lot of cases.

Let me illustrate the case $t = 1/2$:

$u(x,1/2) = 1/2\ [\phi(x-1/2) + \phi(x+1/2)]$

CASE 1: $x < -3/2$

Then $x - 1/2 < -2$ and $x + 1/2 < -1$

In that case $\phi(x-1/2) = 0$ and $\phi(x+1/2) = 0$, so

$u(x,1/2) = 1/2\ (0 + 0) = 0$

(TO BE CONTINUED...)