LECTURE 7: LINEAR TRANSFORMATIONS (II)

Thursday, October 10, 2019 8:07 PM

Welcome to the second part of our LT extravaganza!

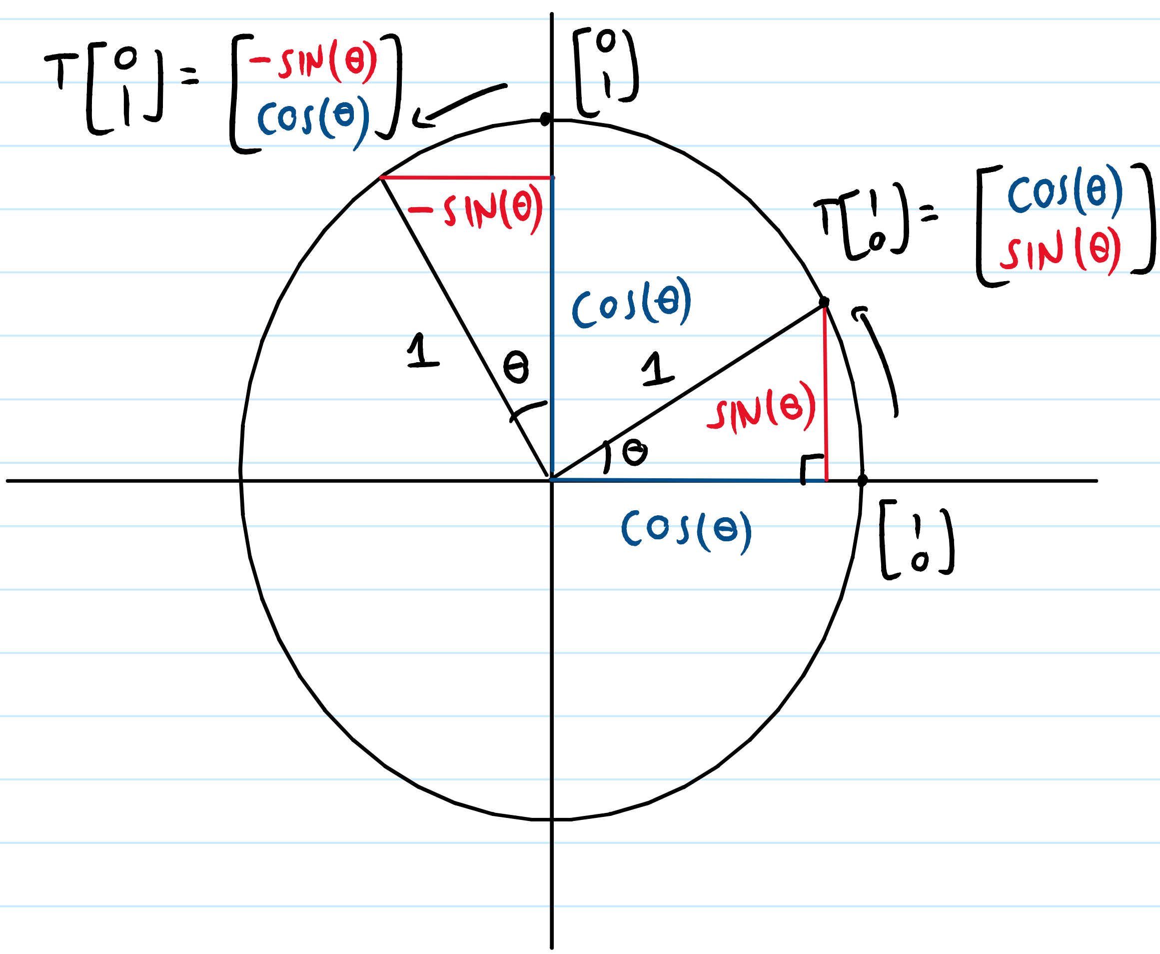
Today: We'll finally see what linear transformations have to do with systems of equations

I- MORE MATRIX PRACTICE

But first, a little bit more practice with matrices

Example:

Find the matrix of T which rotates points in the plane by θ radians (counterclockwise)



$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}}$$

"ROTATION MATRIX"

Mnemonic:

$$\begin{bmatrix}\cos & \xrightarrow{\text{DERIVATIVE}} & -\sin\\ \sin & & \cos\end{bmatrix}$$

(also determinant = 1, see chapter 3)

Example: Find the matrix of T: $R^3 \to R^2$ defined by

$$\underbrace{T\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbb{R}^3} = \underbrace{\begin{bmatrix} x - y + 2z \\ x + y - z \end{bmatrix}}_{\mathbb{R}^2}$$

$$\cancel{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}$$

Consequence: T(x) = Ax, so:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

II- ONTO

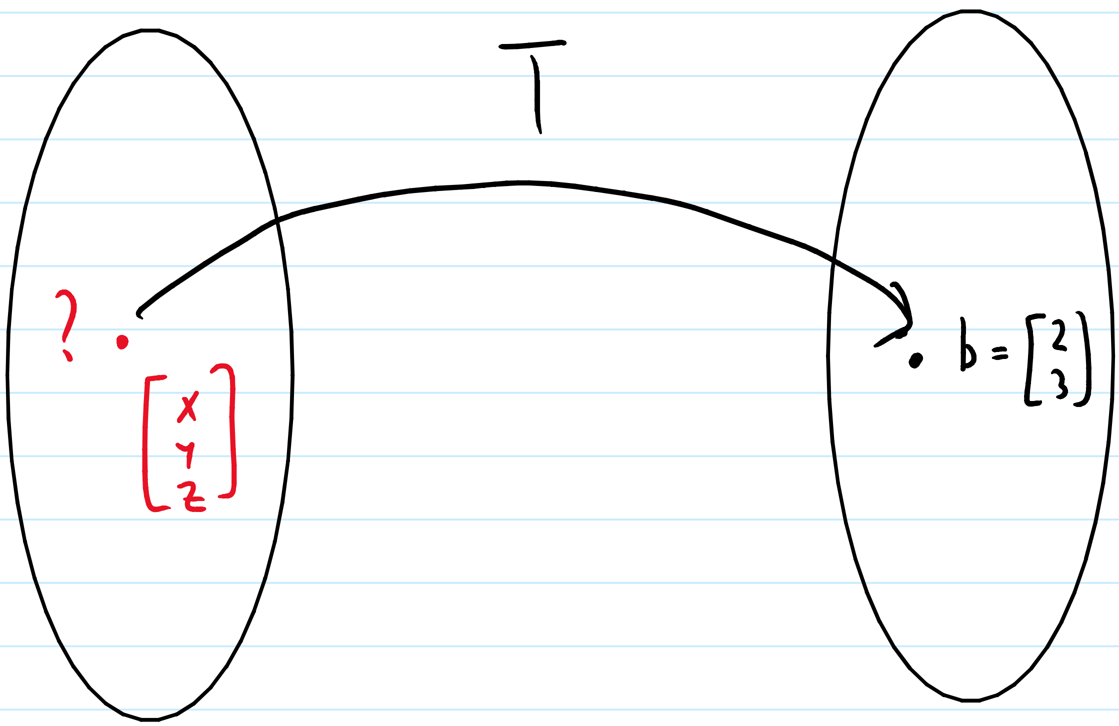
Now let's finally see why this matrix A is so useful

Example: Same T as above

(a) Is **b** = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the RANGE / IMAGE of T ?

Means: Is there $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\underbrace{T\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{A\underline{x}} = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\underline{b}}$?

$$A\underline{x} = \underline{b}$$



Think of it in terms of flights:

If T is an airplane and **b** = Bali, this is saying: Is there a flight to Bali?

EQUIVALENTLY: (This is where A comes in)

Is the following system consistent ?

$$\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\underline{b}}$$

$$(x\text{-}1)\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 1 & 1 & -1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

No row of the form [0 0 0 | *] so YES

(b) Is T onto (R^2) ?

Onto just means you can do this for **every b**:

Definition: T is onto (R^2) if for *every* **b**, *Ax* = **b** is consistent

(Not the actual definition, but the one we'll use in practice)

STUPID WAY: Show for **every b**, *Ax* = b has a solution or not

SMART WAY:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$

Pivot in every **ROW**

=> *Ax* = b is consistent for every **b** (by Row Theorem)

=> T is onto

And in fact, this leads to:

ROW THEOREM DELUXE:

1) Pivot in every row
2) Consistent
3) Span
4) $T(x) = Ax$ is onto R^m (NEW)

ROW
$Ax = b$
SPAN
ONTO

III- ONE-TO-ONE

On the other side of the spectrum, there is the concept of one-to-one:

(c) Is T one-to-one?

Definition: T is one-to-one (1-1) if $Ax = 0 \Rightarrow x = 0$

Here:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Show: $Ax = 0 \Rightarrow x = 0$ (or not)

STUPID WAY: Solve $Ax = 0$

SMART WAY:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$

NOT a pivot in every column
=> Free variables
=> $Ax = 0$ has a nonzero solution
=> NOT 1-1, so NO

And of course, this generalizes to:

COLUMN THEOREM DELUXE:

1) Pivot in every column
2) $Ax = 0 \Rightarrow x = 0$
3) LI
4) $T(x) = Ax$ is one-to-one (NEW)

COLUMN

$Ax = 0$

LI

1-1

Example:

For which h is $T(x) = Ax$ one-to-one? Onto?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & h \end{bmatrix}$$

$$\xrightarrow{REF} \begin{bmatrix} 1 & 2 \\ 0 & h-6 \\ 0 & 0 \end{bmatrix}$$

<u>Onto</u>: There is never a pivot in every row, so never onto

<u>One-to-one</u>: Pivot in every column provided $h - 6 \neq 0$

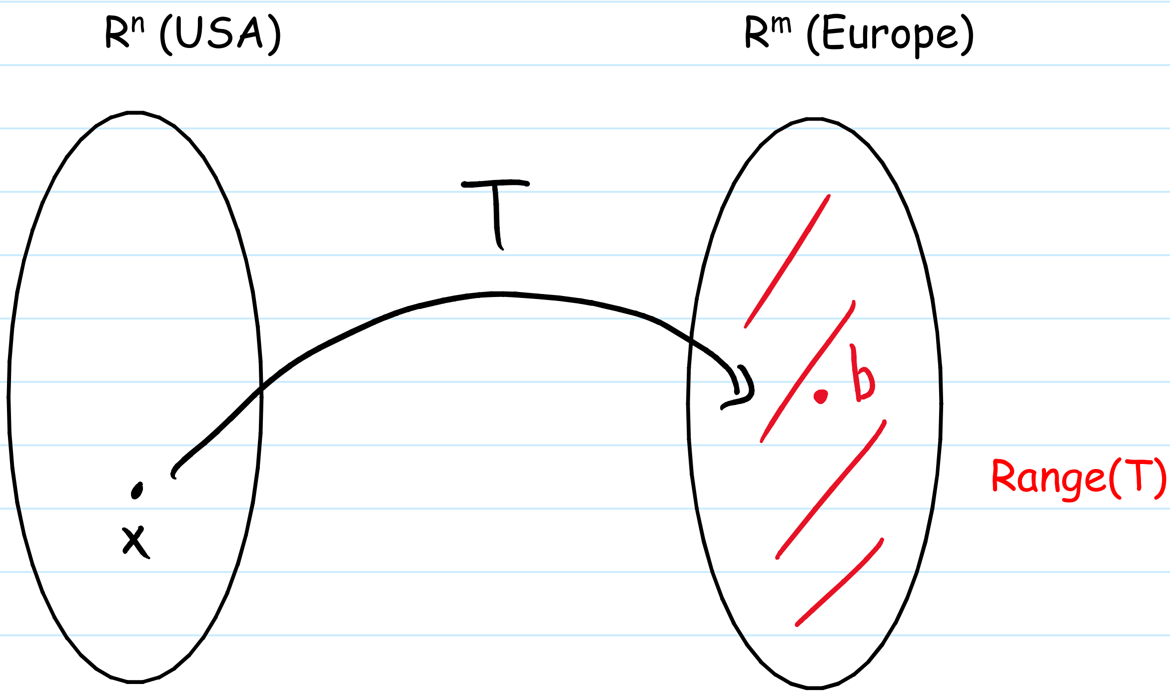
So one-to-one if $h \neq 6$

IV- SOME INTUITION

A - ONTO

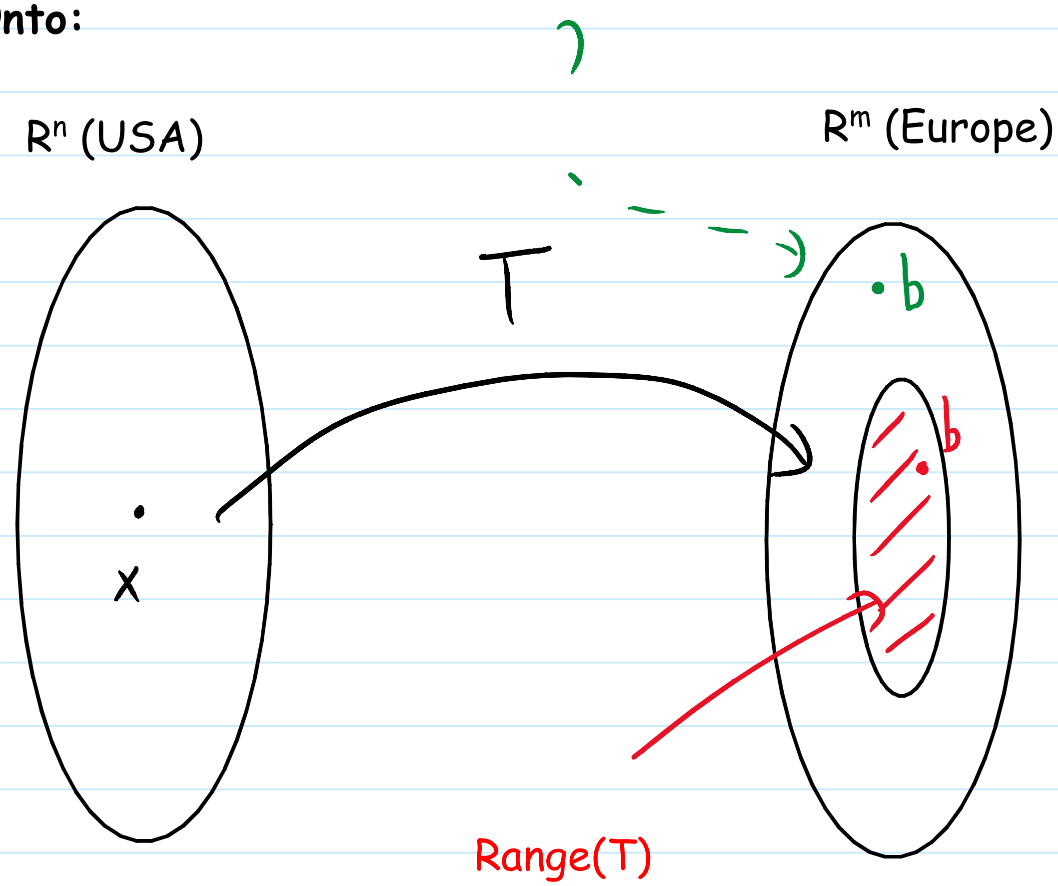
Note: Onto really means that for every b, there is x with $T(x) = b$. Equivalently: $\text{Range}(T) = \mathbb{R}^m$ (the whole space)

Onto:



Think of this in terms of flights: Onto means that you can fly anywhere to Europe from the US. That is, no matter which city b (Basel) in Europe you want to fly to, there's at least one American city x (Xanadu) you can fly from to reach b

Not Onto:



Not onto means that there is some city in Europe (think Stonehenge) which you cannot fly to from the US.

Since we'd like to fly anywhere we want, we like onto functions.

Note:

Do not confuse Codomain (which is just R^m) with Range (which is all the possible values of T = all the b's for which Ax = b is consistent)

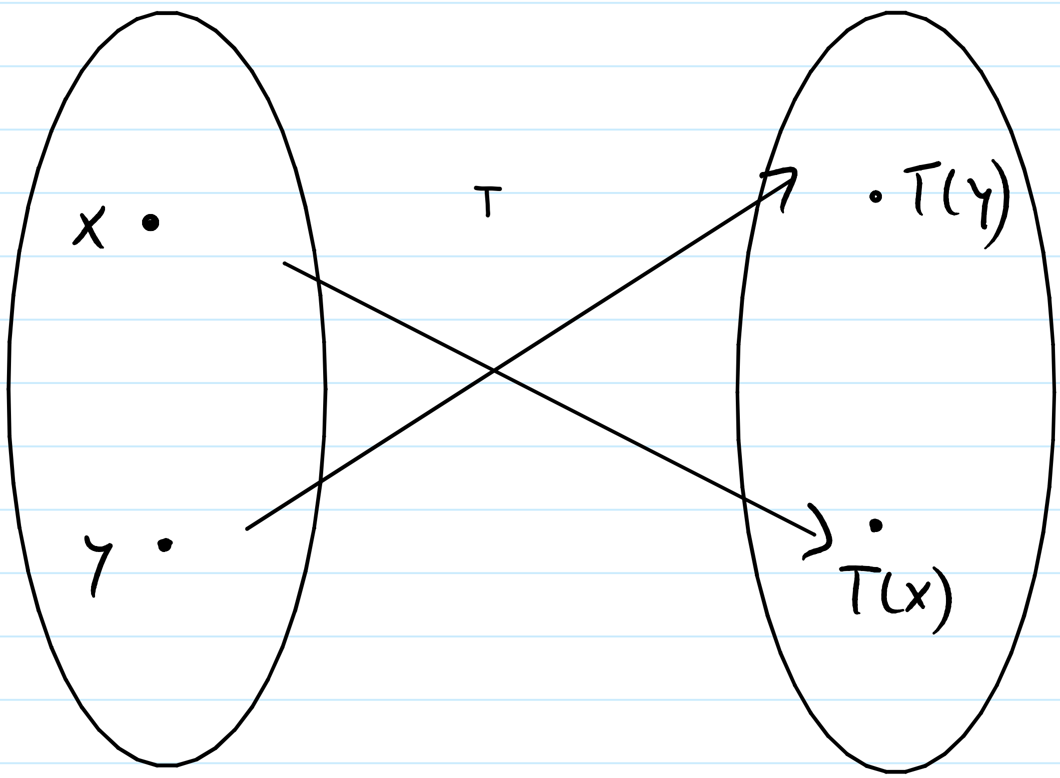
Ex: Here Codomain = Europe, but Range = All the possible cities you can fly to

B- ONE-TO-ONE

T is one-to-one if $x \neq y \Rightarrow T(x) \neq T(y)$

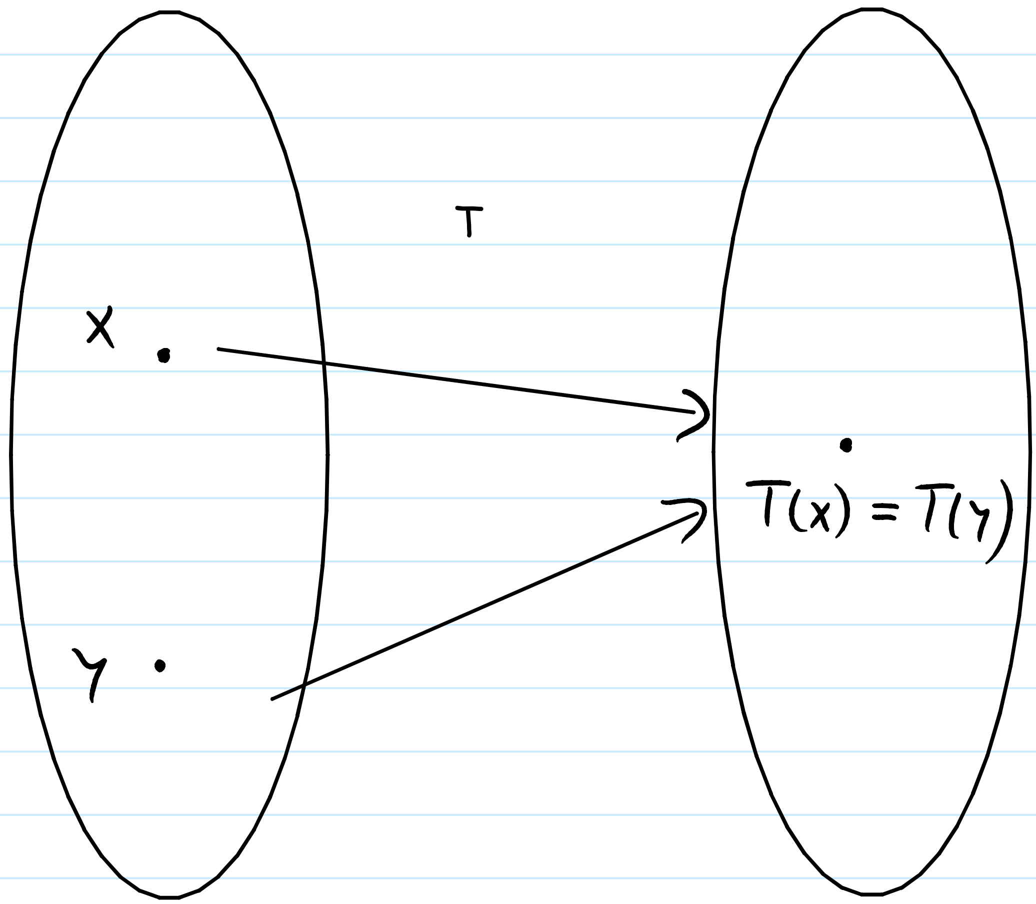
Meaning: Different inputs => Different outputs

One-to-one:



In terms of airplanes: If you depart from a different city, you will land in a different city

Not one-to-one:



Same output, even though different inputs

Airplanes: Two airplanes departing from different cities might land in the same city. This gives a possibility of a crash. Since we don't like crashes, we don't like not one-to-one functions.