LECTURE 8: MATRIX ALGEBRA

Saturday, October 12, 2019   3:10 PM

Welcome to Chapter 2 ! While Chapter 1 focused more on systems of equations, this chapter will focus on matrices. In fact today we're going to cover the matrix... algebra!

I- MATRIX OPERATIONS

Definition: A matrix A is a table of numbers

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ 2x3 matrix

There are lots of things we can do to them:

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

Ex: $(-3)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix}$

Ex: Transpose (= flipping matrix about its diagonal)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(rows become columns, and columns become rows)

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

=> Symmetric matrix

Definition: A is symmetric if $A^T = A$

II- MATRIX MULTIPLICATION

More importantly, we can **multiply** two matrices. In order to achieve this, recall:

Definition: Dot product:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$$

Ex: Calculate AB, where:

$$A = \begin{array}{cc} & N \\ M & \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{cc} & P \\ N & \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Note: Need # of columns of A = # of rows of B

(You don't actually need to check this; this is something you would notice anyway when calculating AB)

Then AB will be m x p

Mnemonic: $\underbrace{(m \times \cancel{n})}_{A} \cdot \underbrace{(\cancel{n} \times p)}_{B} = \underbrace{m \times p}_{AB}$

Here: AB will be 2 x 2

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 1 & 4 \end{bmatrix}$$

$1 \times 2 + 0 \times 1 + 2 \times 3$

$1 \times 0 + 0 \times 1 + 2 \times 4$

$(-1) \times 2 + 0 \times 1 + 1 \times 3$

Then matrix multiplication is just a bunch of dot products!

First: you take the <u>first</u> row of A and the <u>first</u> column of B to dot it.

Then: you still take the first row of A, but this time the second column of B, and dot it.

Now since you ran out of columns, you move on to the second row of A (and the first column of B), and repeat!

It should remind you of FOIL-ing out an expression

Ex: Calculate AB and BA, where:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 10 & 5 & 10 \\ 16 & 8 & 16 \end{bmatrix}$$

WARNING: In general $AB \neq BA$!!!

Also: $AB = AC \not\Rightarrow B = C$!

(Basically, matrices are weird)

Ex: Let $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+2z \\ x+y-z \end{bmatrix}$

Found: $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

Then: $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \boxed{1 \quad -1 \quad 2} \\ 1 \quad\ \ 1 \quad -1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} x \\ y \\ z \end{matrix}} \end{bmatrix}$

$$= \begin{bmatrix} x-y+2z \\ x+y-z \end{bmatrix}$$

$$= T\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

THIS is why $T(x) = Ax$

Ex: Identity matrix

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2×2) OR $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (3×3)

FACT: $AI = IA = A$ for every A

Analog of 1 in the matrix world, SUPER important even though it looks innocent!

III- MATRIX POWERS

Ex: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Calculate:

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = AAA = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

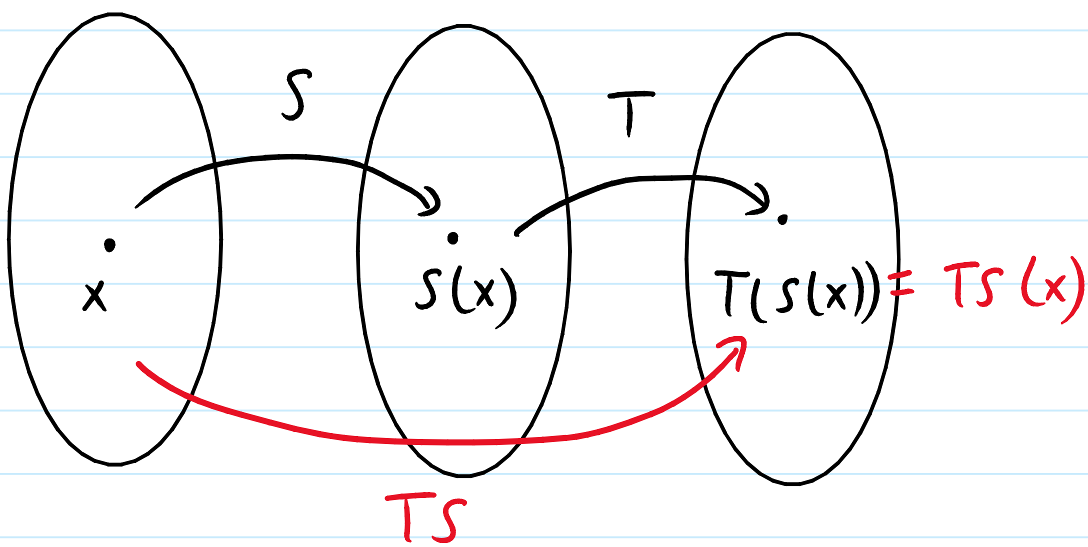
$$A^4 = A^3 A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

In general: $A^n = AAA \ldots A$ (n times) $= \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}$

Note: In chapter 5, we'll find an EASY way of calculating A^n

IV- INTERPRETATION OF MATRIX MULTIPLICATION

Why is matrix multiplication so weird? It's in order for the following fact to hold:



Definition: If S and T are LT, then the composition TS (or ToS) is defined by:

$$TS(x) = T(S(x))$$

If you think of S and T as flights, then TS is a direct flight which brings you directly from x to T(S(x))

Fact: If the matrix of T is A and the matrix of S is B, then the matrix of TS is AB

V- THE INVERSE OF A MATRIX (Section 2.2)

Just as we defined AB, we can define 1/A, or A^{-1}

Definition: A^{-1} is the matrix B such that AB = BA = I, that is:

$$\boxed{AA^{-1} = A^{-1}A = I}$$

(Think: A (1/A) = (1/A) A = 1, analog of 1/x in the matrix world)

Definition: A is invertible if such a matrix B exists.

Ex: $[7]^{-1} = [1/7]$ (1 x 1 matrix)

Ex:

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1}$$

Fact:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{Some number}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

("determinant")

Basically, you flip the diagonal terms, and put a minus on the other terms.

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{\underbrace{1\times 4 - 1\times 3}_{1}} \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

WARNING: This trick only works for 2 x 2 matrices!

Will do larger matrices next time.

Why useful?

AMAZING FACT:

$$Ax = b \Rightarrow x = A^{-1} b$$

So this gives us a 1 second way of solving systems, PROVIDED A^{-1} EXISTS !!!

Ex: Solve:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\underline{b}}$$

$$\underline{x} = A^{-1} \underline{b} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{(by above)}$$

$$= \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

So if A^{-1} exists, everything is awesome, which begs the question: **WHEN** does A^{-1} exist? (which we'll answer next time)