LECTURE 8: ENERGY METHODS

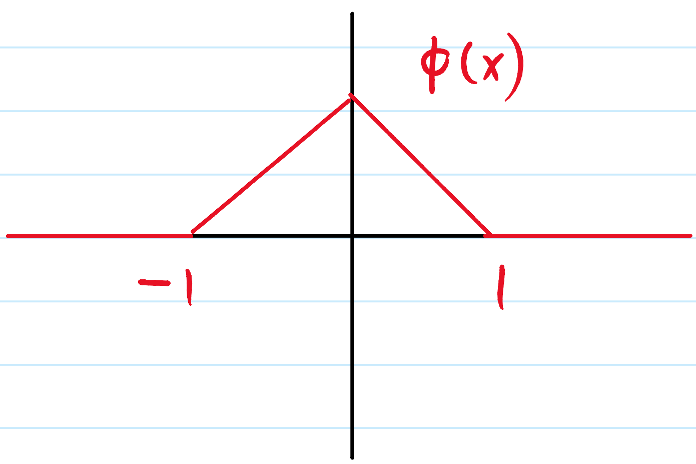
Friday, October 11, 2019 5:42 PM

I- THE PLUCKED STRING (section 2.1)

Example 2: [The plucked string]

$u_{tt} = u_{xx}$ ($c = 1$) with $u(x,0) = \phi(x)$ and $u_t(x,0) = 0$, where:

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ 1 - |x| & \text{for } -1 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$



D'Alembert says:

$u(x,t) = 1/2\ [\ \phi(x-t) + \phi(x+t)\ ]$

Now given the piecewise definition of ϕ, this becomes quite complicated, and we need to split this up into a lot of cases.

Let me illustrate the case $t = 1/2$:

$u(x,1/2) = 1/2\ [\phi(x-1/2) + \phi(x+1/2)]$

CASE 1: $x < -3/2$

Then $x - 1/2 < -2$ and $x + 1/2 < -1$

In that case $\phi(x-1/2) = 0$ and $\phi(x+1/2) = 0$, so

$u(x,1/2) = 1/2\ (0 + 0) = 0$

CASE 2: $-3/2 < x < -1/2$

Then $x - 1/2 < -1$ but $-1 < x + 1/2 < 0$

In that case $\phi(x-1/2) = 0$ but

$\phi(x+1/2) = 1 - |x+1/2| = 1 -(-x-1/2) = 3/2 + x$

So $u(x,1/2) = 1/2(0 + 3/2 + x) = 3/4 + x/2$

CASE 3: $-1/2 < x < 1/2$

Then $-1 < x - 1/2 < 0$ but $0 < x + 1/2 < 1$

In that case:

$\phi(x-1/2) = 1 - |x-1/2| = 1 + x - 1/2 = x + 1/2$

$\phi(x+1/2) = 1 - |x+1/2| = 1 -(x+1/2) = 1/2 - x$

But then

$u(x,1/2) = 1/2\ (x + 1/2 + 1/2 - x) = 1/2\ (1) = 1/2$

CASE 4: **$1/2 < x < 3/2$**

Then $0 < x - 1/2 < 1$ and $1 < x + 1/2 < 2$

$\phi(x-1/2) = 1 - |x-1/2| = 1-(x-1/2) = 3/2 - x$
$\phi(x+1/2) = 0$

So $u(x,1/2) = 1/2\ (3/2 - x) = 3/4 - x/2$

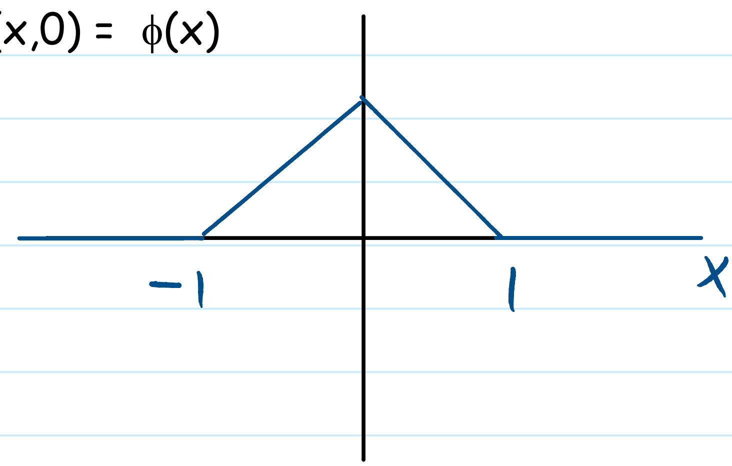
CASE 5: **$x > 3/2$**

Then $x - 1/2 > 1$ and $x + 1/2 > 2 > 1$, so

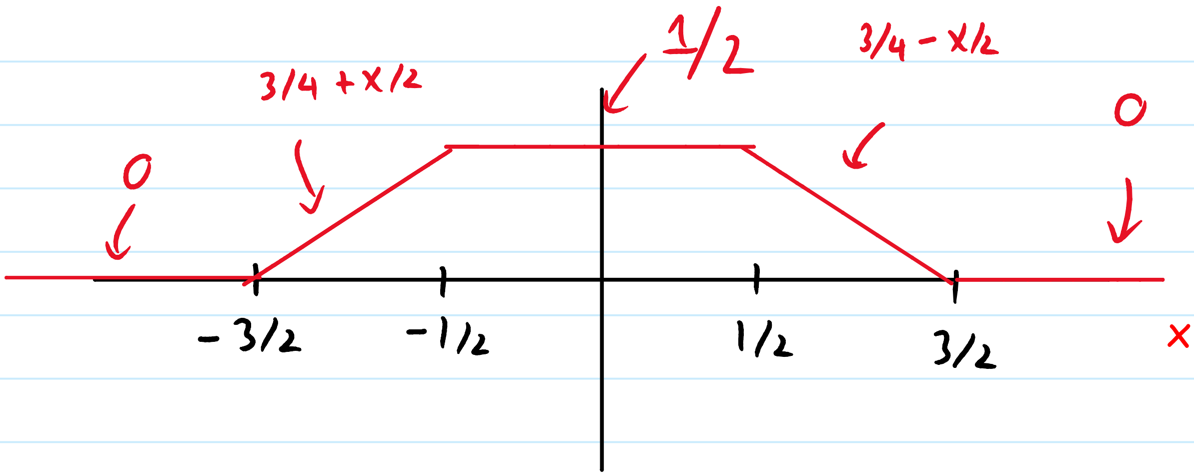
$\phi(x-1/2) = 0$ and $\phi(x+1/2) = 0$, and so $u(x,1/2) = 0$

Picture: $u(x,t)$ for various t

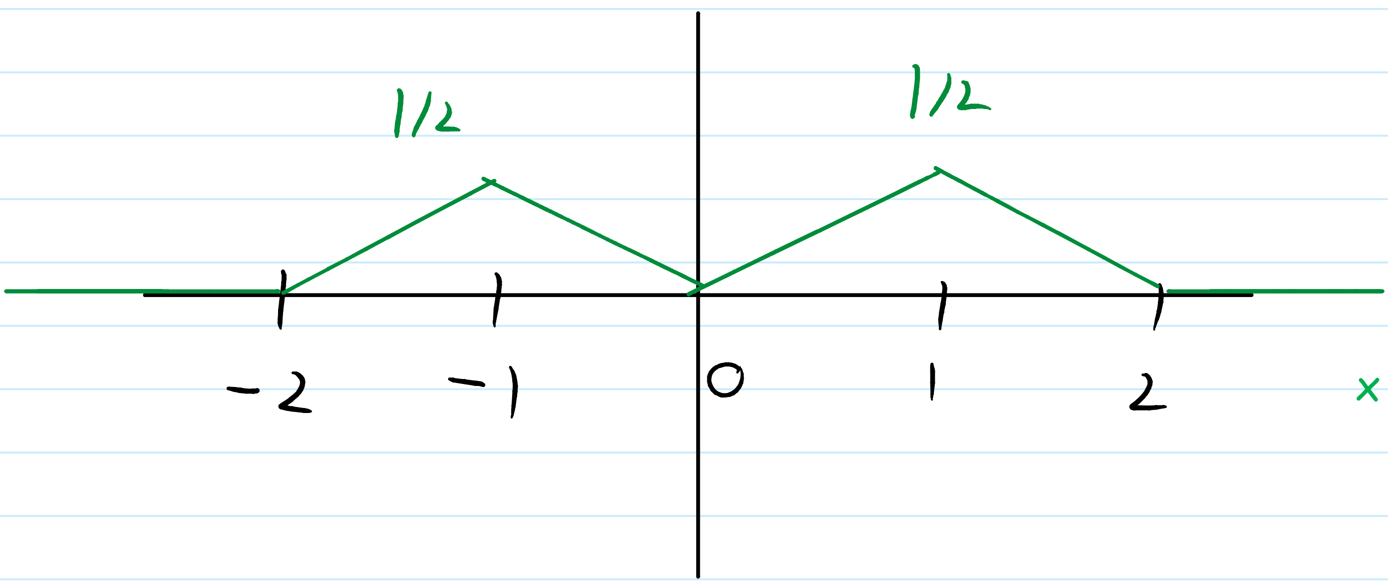
$t = 0$ $u(x,0) = \phi(x)$



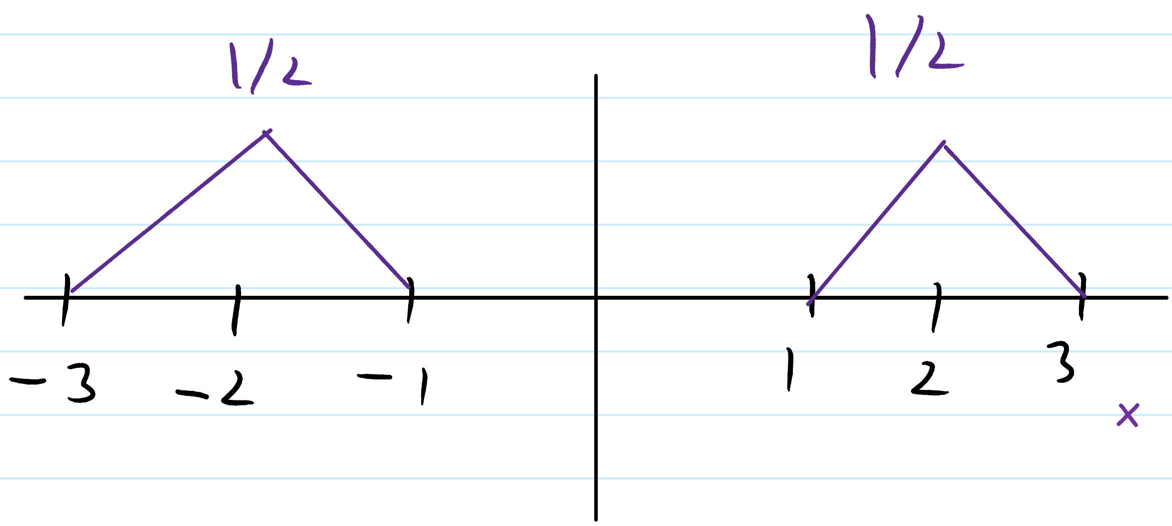
t = 1/2 Discussed above



t = 1 HW



$t = 2$



And so on and so forth!

This gives us a nice "movie" of the solution

II- THE ENERGY METHOD (Section 2.2)

Let's continue by discussing some more general properties of the wave equation.

Note: Everything below *any* solution of the wave equation. We are **NOT** using d'Alembert's formula here!

There are two main classes of PDE methods: Maximum Principle Methods (based on the maximum principle in 2.3) and Energy Methods (based on integration by parts).

Here, let me illustrate how the energy method works:

MAIN RESULT: [CONSERVATION OF ENERGY]

Suppose u solves $u_{tt} = u_{xx}$

Then the following energy E(t) is conserved:

$$E(t) = 1/2 \int_{-\infty}^{\infty} (u_t)^2 + (u_x)^2 \, dx$$

Note: In physics, the first term is called the kinetic energy ($1/2\ m v^2$) and the second part the potential energy, so this says that the total energy is conserved.

Method 1: Show $E'(t) = 0$

Could go that route, and it's in fact easier, but it requires you beforehand to know what E is!

Method 2: Energy Method

Start with:

$$u_{tt} = u_{xx}$$

Multiply both sides by u_t

$$u_{tt} u_t = u_{xx} u_t$$

Now integrate with respect to x:

$$\underbrace{\int_{-\infty}^{\infty} u_{tt}\, u_t \, dx}_{A} = \underbrace{\int_{-\infty}^{\infty} u_{xx}\, u_t \, dx}_{B}$$

STUDY OF A:

Note that from calculus: $y'' y' = \frac{1}{2} \left[(y')^2 \right]'$

Hence: $u_{tt} u_t = \frac{1}{2} \frac{d}{dt} (u_t)^2$

In particular:

$$A = \int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dt} (u_t)^2 \, dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} (u_t)^2 \, dx$$

STUDY OF B:

Here we integrate by parts with respect to x

Note:

1) Integration by parts: $\int f'g = fg - \int fg'$

2) Here we assume the fg term is 0 (which basically means that our waves are 0 at x = +/- infinity, which makes sense in practice)

3) y y' = 1/2 $(y^2)'$

$$\int_{-\infty}^{\infty} U_{xx} U_t \, dx \overset{\text{IBP}}{=} -\int_{-\infty}^{\infty} U_x U_{xt} \, dx$$

$$= -\int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dt} (U_x)^2 dx$$

$$= -\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} (U_x)^2 dx$$

A = B then implies:

$$\frac{1}{2} \frac{d}{dt} \int^{\infty} (U_t)^2 = -\frac{1}{2} \frac{d}{dt} \int^{\infty} (U_x)^2 dx$$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty} (U_t) = -\frac{1}{2}\frac{d}{dt}\int_{-\infty} (U_x) \, dx$$

$$\frac{d}{dt}\left[\underbrace{\frac{1}{2}\int_{-\infty}^{\infty} (U_t)^2 + (U_x)^2 \, dx}_{E(t)}\right] = 0$$

Therefore $E(t)$ is constant!

Remarks:

1) In particular $E(t) = E(0)$ and $E(0)$ only depends on our initial conditions ϕ and ψ

2) **Application:** Uniqueness of solutions

   Suppose u and v both solve the wave equation

   Let $w = u-v$, then w also solves the wave equation (check) but with $\phi = 0$ and $\psi = 0$ (check)

   Then, by # 1 in 2.2 (on HW), get $w = 0$, so $u-v = 0$ so $u = v$