# LECTURE 9: THE HEAT EQUATION (I)

Monday, October 14, 2019 5:23 PM

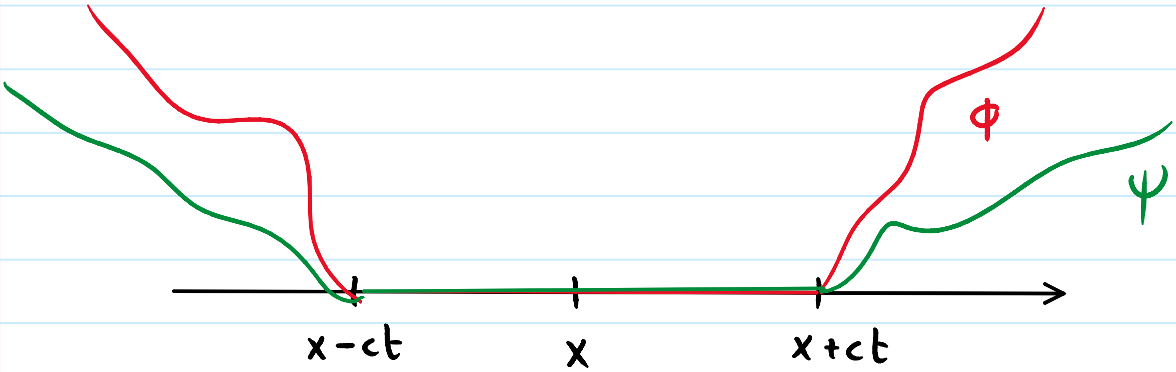
## I- DOMAIN OF DEPENDENCE (section 2.2)

Let's look again at D'Alembert's formula:

$$u(x,t) = 1/2\,(\phi(x-ct) + \phi(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds$$

Notice that the value of u at (x,t) only depends on the values of $\phi$ and $\psi$ on the interval [x-ct,x+ct].

In particular, suppose $\phi = 0$ and $\psi = 0$ on the interval [x-ct,x+ct] but are nonzero elsewhere, as in this picture:

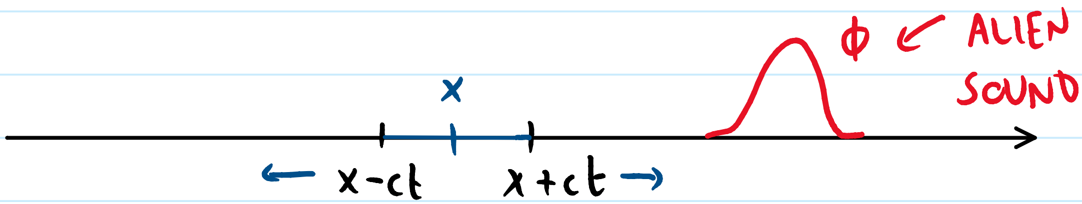



Then u(x,t) = 0

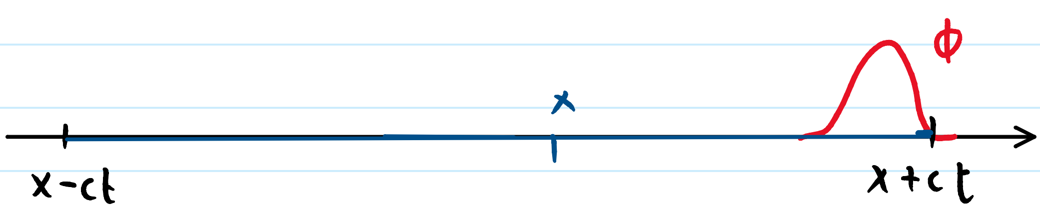
In other words, anything **outside** the interval [x-ct,x+ct] doesn't affect u at all!

**Interpretation:** The wave equation has **finite** speed of propagation: If an alien far far away (outside the interval [x-ct,x+ct]) yells at you, you won't really hear the sound until time is so large that $\phi$ or $\psi$ is nonzero on [x-ct,x+ct].

Ex: t small: Here u(x,t) = 0



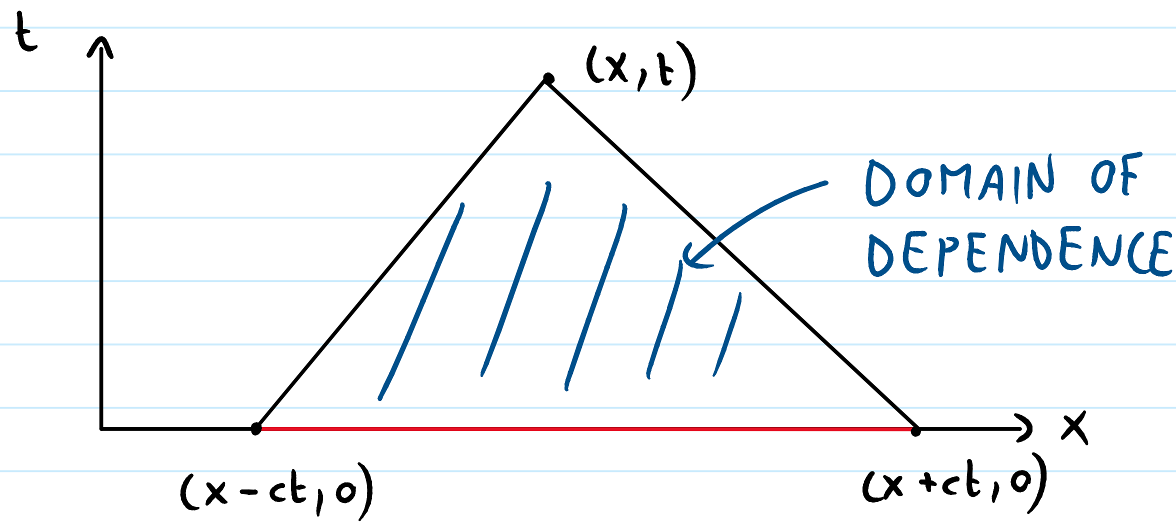

t large, u(x,t) is nonzero:




We can represent this pictorially as follows:

Start with (x,t) and connect it with the points (x-ct,0) and (x+ct,0) (Why? $\phi(x-ct) = u(x-ct,0)$)

**Picture:**




You get a triangle called the "domain of dependence"

**Interpretation:** If $\phi$ and $\psi$ are both 0 on the bottom leg of the triangle, then u = 0 at the top vertex (and even on the whole triangle!)

**Note:** In higher dimensions, the triangle becomes a cone.

This ends our 'first look' at the wave equation! We'll come back to it very soon, but now let's discuss our second important PDE: The Heat Equation!

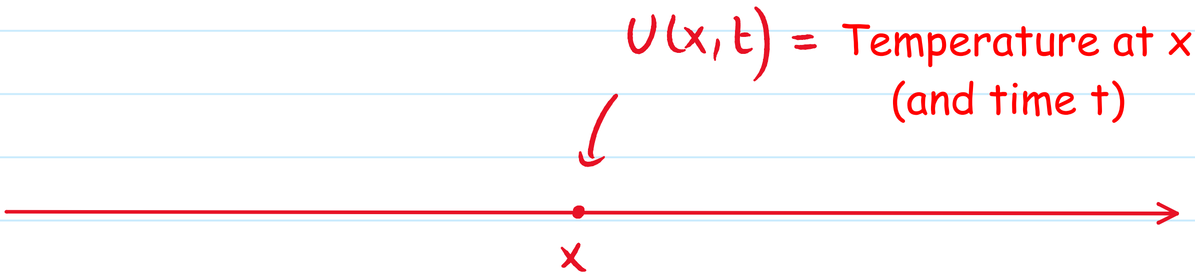
## II- THE HEAT EQUATION

$u = u(x,t)$

$u_t = k\, u_{xx}$
$u(x,0) = \phi(x)$

Also known as the diffusion equation, $u(x,t)$ measures the temperature of an (infinite) metal rod at position $x$ and time $t$

**Picture:** Time = $t$




## III- FUNDAMENTAL SOLUTION (Section 2.4)

Let's first focus on finding a solution (section 2.4), then then move on to discussing general properties (section 2.3).

WARNING: Our approach is VERY different from the wave equation. Here, instead of finding the general solution, we will first find **one** solution, and then build the general solution from that.

**Goal:** Find a solution of $u_t = u_{xx}$ $(k = 1)$

In this case, we will look for a solution of a special form

## STEP 0

Silly trick: If you ignore u in $u_t = u_{xx}$ you get:

$$t = x^2$$

$$\Rightarrow \quad \frac{x^2}{t} = 1$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{t}}\right)^2 = 1$$

So it might seem like the quantity $y = \dfrac{x}{\sqrt{t}}$

might be useful, and indeed it is!

## STEP 1

Guess: $u(x,t) = v\left(\dfrac{x}{\sqrt{t}}\right)$

For some function $v = v(y)$ to be found

This doesn't *quite* work, unfortunately.

Better guess:

$$u(x,t) = \frac{1}{t^{\alpha}}\, v\left(\frac{x}{\sqrt{t}}\right)$$

$$\boxed{u(x,t) = t^{-\alpha}\, v\left(x\, t^{-1/2}\right)}$$

For some $\alpha$ and v to be determined.

(Intuitively: u "blows up" near t = 0)

## STEP 2

Plug our guess in our PDE $u_t = u_{xx}$

$$u_x = \left(t^{-\alpha}\, v\left(x\, t^{-1/2}\right)\right)_x$$

$$= t^{-\alpha}\, v'\left(x\, t^{-1/2}\right)\left(t^{-1/2}\right)$$

$$= t^{-\alpha - \frac{1}{2}}\, v'\left(x\, t^{-1/2}\right)$$

$$u_{xx} = t^{-\alpha - \frac{1}{2}}\, v''\left(x\, t^{-1/2}\right)\left(t^{-1/2}\right)$$

$$= t^{-\alpha - 1}\, v''(y)$$

$$u_t = \left(t^{-\alpha}\, v\left(x\, t^{-1/2}\right)\right)_t$$

$$= -\alpha\, t^{-\alpha - 1}\, v\left(\frac{x}{\sqrt{t}}\right) + t^{-\alpha}\, v'\left(\frac{x}{\sqrt{t}}\right)\left(x\, t^{-1/2}\right)_t$$

$$= -\alpha t^{-\alpha-1} v(y) + t^{-\alpha} v'(y) \left(-\frac{x}{2} t^{-3/2}\right)$$

$$= -\alpha t^{-\alpha-1} v(y) - \frac{x}{2} t^{-\alpha-\frac{3}{2}} v'(y)$$

So $u_t = u_{xx}$ implies:

$$-\alpha \cancel{t^{-\alpha-1}} v - \frac{x}{2} \cancel{t^{-\alpha-\frac{3}{2}}} v' = \cancel{t^{-\alpha-1}} v''$$

$$-\frac{\alpha}{\cancel{t}} v - \frac{x}{2\cancel{t}\sqrt{t}} v' = \frac{1}{\cancel{t}} v''$$

$$-\alpha v - \underbrace{\frac{\left(\frac{x}{\sqrt{t}}\right)}{2}}_{"y"} v' = v''$$

$$\Rightarrow \quad v'' + \frac{y}{2} v' + \alpha v = 0$$

**Note:** If we didn't include the $1/t^{\alpha}$ term, we wouldn't have such a nice simplification with the y (and instead have weird x and t terms)

At this point we are stuck, but remember that we still have a choice in our constant $\alpha$ !

## STEP 3

Choose $\alpha = 1/2$ (this allows us to factor out the second and third term)

Then we get:

$$V'' + \frac{y}{2} V' + \frac{1}{2} V = 0$$

$$V'' + \frac{1}{2} (y V' + V) = 0$$

$$V'' + \frac{1}{2} (y V)' = 0$$

$$\left[ V' + \frac{1}{2} y V \right]' = 0$$

$$\Rightarrow \quad V' + \frac{y}{2} V = C$$

**Assume** $C = 0$

(**Why?** We just need to find **ONE** solution. Moreover, in physical models, v(y) and v'(y) goes to 0 as y goes to infinity)

$$\Rightarrow V' + \frac{y}{2} V = 0$$

STEP 4 Solve this!

$$v' = -\frac{y}{2} v$$

$$\Rightarrow \frac{v'}{v} = -\frac{y}{2}$$

$$\Rightarrow \left[\ln|v|\right]' = -\frac{y}{2}$$

$$\Rightarrow \ln|v| = -\frac{y^2}{4} + C$$

$$\Rightarrow |v| = e^{-\frac{y^2}{4} + C}$$

$$\Rightarrow v = \underbrace{\pm e^{C}}_{\text{Arbitrary}} e^{-\frac{y^2}{4}}$$

=> $v(y) = C e^{-\frac{y^2}{4}}$

STEP 5: CONCLUSION

$$u(x,t) = \frac{1}{t^{1/2}} v\left(\frac{x}{\sqrt{t}}\right) \qquad \left(\alpha = \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{t}} C e^{-\frac{1}{4}\left(\frac{x}{\sqrt{t}}\right)^2}$$

$$u(x,t) = \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

Finally, choose $C = \frac{1}{\sqrt{4\pi}}$ (see below why), then

$$\frac{C}{\sqrt{t}} = \frac{\frac{1}{\sqrt{4\pi}}}{\sqrt{t}} = \frac{1}{\sqrt{4\pi t}}$$

**Definition:** The fundamental solution $S(x,t)$ of the heat equation $u_t = u_{xx}$ is:

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

**Remarks:**

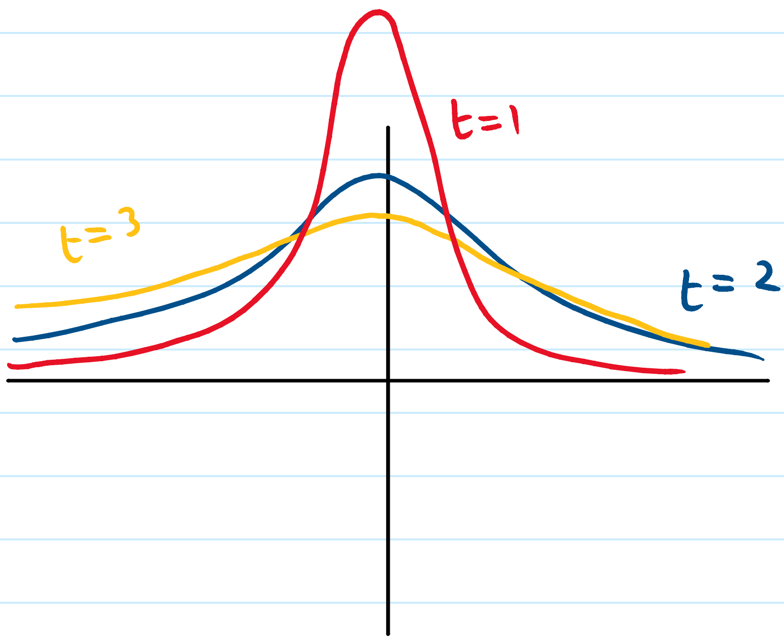
1) The reason we chose $C$ as we did is because, with this choice of $C$, we have:

$$\int_{-\infty}^{\infty} S(x,t)\, dx = 1$$

That is, for each t, the area under the solution is 1 (see # 6 and 7 in 2.4 on HW # 4)

2) $S(x,t)$ looks like the bell curve $\exp(-x^2)$, but more and more spread out as t gets larger. At each time, the area under the curve is 1.

**Picture:**




3) To solve $u_t = k\, u_{xx}$ replace t by kt to get that the fundamental solution is:

$$\boxed{S(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{x^2}{4kt}}}$$

This is the version we'll be dealing with throughout the course.