MIDTERM SOLUTIONS

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PROBLEM 1

$$y^2 x u_x + x^3 y u_y = 0$$

1) $\frac{dy}{dx}$ = Slope = $\frac{x^3 y}{y^2 x} = \frac{x^2}{y}$

2) $y\,dy = x^2\,dx$

$$y\,dy = x^2\,dx$$

$$1/2\, y^2 = 1/3\, x^3 + C$$

$$\underbrace{1/2\, y^2 - 1/3\, x^3}_{?} = C$$

3) $u(x,y) = f(?)$

$$\boxed{u(x,y) = f(1/2\, y^2 - 1/3\, x^3)}$$

PROBLEM 2

$$u(x,t) = F(4x + t) + G(x-t)$$

1) $u(x,0) = F(4x + 0) + G(x-0) = F(4x) + G(x) = e^x$

$F(4x) + G(x) = e^x$

2) $u_t(x,t) = (F(4x + t)+G(x-t))_t$
$= F'(4x+t) - G'(x-t)$

$u_t(x,0) = F'(4x + 0) - G'(x-0) = F'(4x) - G'(x) = \cos(x)$

$F'(4x) - G'(x) = \cos(x)$

$\int F'(4s) - G'(s) \, ds = \int \cos(s) \, ds$

$1/4 \, F(4x) - G(x) = \sin(x) + C$

(Careful: An antiderivative of $F'(4s)$ with respect to s is NOT $F(4s)$, but $1/4 \, F(4s)$. Here don't confuse $F'(4s)$ with $(F(4s))'$)

3) Hence we have:

$$F(4x) + G(x) = e^x$$
$$1/4 \, F(4x) - G(x) = \sin(x) + C$$

Add both equations:

$5/4 \, F(4x) = e^x + \sin(x) + C$

$F(4x) = 4/5 \, e^x + 4/5 \sin(x) + 4/5 \, C$

In particular (replacing 4x with x, so dividing every input by 4)

$F(x) = 4/5\, e^{x/4} + 4/5 \sin(x/4) + 4/5\, C$

Subtract 4 times the second equation from the first one:

$5\, G(x) = e^x - 4 \sin(x) - 4C$

$G(x) = 1/5\, e^x - 4/5 \sin(x) - 4/5\, C$

4) Finally:

$$
\begin{aligned}
u(x,t) &= F(4x+t) + G(x-t) \\
&= 4/5\, e^{(4x+t)/4} + 4/5 \sin((4x+t)/4) + 4/5\, C \\
&\quad + 1/5\, e^{x-t} - 4/5 \sin(x-t) - 4/5\, C
\end{aligned}
$$

$$\boxed{u(x,t) = 4/5\, e^{(4x+t)/4} + 1/5\, e^{x-t} + 4/5 \sin((4x+t)/4) - 4/5 \sin(x-t)}$$

(No need to simplify this)

PROBLEM 3

1) Let $w = u - v$

Then $w_t = (u-v)_t = u_t - v_t = k u_{xx} - k v_{xx} = k\,(u-v)_{xx} = k\, w_{xx}$

So w also solves the heat equation

2) By the Maximum principle, max w is the larger one of

$\max w(x,0)$, $\max w(0,t)$, $\max w(l,t)$

But $w(x,0) = u(x,0) - v(x-0) \leq 0$ (by assumption), so $\max w(x,0) \leq 0$

And similarly $\max w(0,t) \leq 0$ and $\max w(l,t) \leq 0$

Therefore, the larger one of those 3 must still be ≤ 0

So $\max w \leq 0$

3) So $w(x,t) \leq 0$ for all x and t

So $u - v \leq 0$

Hence $\boxed{u \leq v}$

PROBLEM 4

1) Multiply both sides of the PDE by u:

$$u_t\, u = k\, u_{xx}\, u - u^3\, u$$

And integrate with respect to x from 0 to l:

$$\int_0^{\ell} u_t\, u\, dx = k \int_0^{\ell} u_{xx}\, u\, dx - \int_0^{\ell} u^4\, dx$$

2) Since $u_t\, u = 1/2\ d/dt\ (u^2)$, the first term just becomes

$$1/2 \int_0^{\ell} d/dt\ u^2\, dx = d/dt\ 1/2 \int_0^{\ell} u^2\, dx$$

3) Integrating the second term by parts, we get

$$u_{xx}\, u\, dx = u_x(l,t)\, u(l,t) - u_x(0,t)\, u(0,t) - \int_0^{\ell} u_x\, u_x\, dx$$

$$= \cancel{u_x(l,t)\, u(l,t)} - \cancel{u_x(l,t)\, u(l,t)} - \int_0^{\ell} (u_x)^2\, dx \quad \text{(by assumption)}$$

$$= -\int_0^{\ell} (u_x)^2\, dx$$

$$d/dt\ 1/2 \int_0^{\ell} u^2\, dx = \underbrace{-k \int_0^{\ell} (u_x)^2\, dx - \int_0^{\ell} u^4\, dx}_{\leq 0} \leq 0$$

$E'(t) \leq 0$

4) Hence the energy $E(t) = 1/2 \int_0^{\ell} u^2(x,t)\, dx$ is decreasing, so

$E(t) \leq E(0)$

$$(0 \leq)\ 1/2 \int_0^{\ell} u^2(x,t)\, dx \leq 1/2 \int_0^{\ell} u^2(x,0)\, dx = 0 \text{ (by assumption)}$$

Hence $1/2 \int_0^{\ell} \underbrace{u^2(x,t)}_{\geq 0}\, dx = 0$

So $u^2(x,t) = 0$, so $u(x,t) = 0$, as we wanted to show.