MATH 112A — MIDTERM STUDY GUIDE

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GENERAL INFO

The Midterm Exam takes place on Wednesday, October 30, during the usual time (4 – 4:50 pm) and the usual place (1600 DBH). I’ll probably give you 5 minutes extra, so make sure you can stay until 4:55 pm. Please bring your student ID, as we’ll be checking IDs during the exam. There will be a seating chart for the exam, which I’ll send out a day or two before the exam.

This is the study guide for the exam. Please read it carefully, because it contains a lot of info about what’s going to be on the exam and what I expect you to know or not to know. That said, remember that this study guide is just a guide and not a complete list. I’ve tried to make this list as complete as possible, but there are always things that I may have missed.

The midterm covers sections 1.1 – 1.2, 1.6, and 2.1 – 2.4 inclusive.

There will be 4 problems on the exam. They will be based on the lecture notes, the homework, and the practice exam. There is no need to read the book. You don’t need to study the quizzes or the discussion worksheets, since I didn’t look at them at all.

Finally, remember that I’m on your side: This class will be curved, in the sense that I will take all your raw scores, add them up, and then curve that total raw score. So the harder the midterm, the more generous the curve. I won’t curve individual exams. You can find the official grading percentages on the syllabus. Also, remember that your final exam score can replace your midterm score if you do better on the final, so this exam should just be a freebie!

Date: Wednesday, October 30, 2019.
THINGS YOU ABSOLUTELY NEED TO KNOW

Here is the **BARE MINIMUM** of what I expect you to know. There will certainly be more than that on the exam!

1. Derive the solution of $au_x + bu_y = 0$
2. Find the solution of more general first-order PDE like $xu_x + yu_y = 0$
3. The Coordinate Method (I will give you the coordinates)
4. Find the general solution of $u_{tt} = c^2 u_{xx}$, using the Factoring Method and the Coordinate Method.
5. Derive d’Alembert’s Formula
6. Do (4) and (5) with more general PDE like and $u_{xx} - 3u_{xt} - 4u_{tt} = 0$
7. You do NOT need to memorize the formula for $S(x, t)$, but you need to know how to derive it, starting from the point: Suppose $u(x, t) = \frac{1}{\sqrt{t}} v \left( \frac{x}{\sqrt{t}} \right)$ and plug this into $u_t = ku_{xx}$. You don’t need to find the constant C.
8. Energy methods, both for the wave and the heat equation, (I would give you the function to multiply the PDE with) and deriving uniqueness from that
9. Maximum principle (for the heat equation), and deriving uniqueness from that.
10. Solve PDEs by changing functions, things like Let $v = e^{-u}$, what PDE does $v$ solve? See 2.4.16 for example.

**Note:** 1.1.11 means Problem 11 in section 1.1.

**Section 1.1: What is a Partial Differential Equation**

- Solve some simple PDE like $u_x = 0$, $u_{xx} = 0$, $u_{xy} = 0$
- Determine if if a PDE is linear or not (1.1.2, 1.1.3)
- Find the order of a PDE, and determine if it’s homogeneous or not
- Know the fact that the general solution of $L(u) = f$ is equal to the general solution of $L(u) = 0 +$ a particular solution of $L(u) = f$
- Verify if a function solves a PDE (1.1.11, 1.1.12)

**Section 1.2: First-Order Linear Equations**

- Find the solution of $au_x + bu_y = 0$ using the geometric method
- In particular, know how to solve the transport equation $u_t + cu_x = 0$
- Do the same thing with the coordinate method. In that case, I would give you the coordinates beforehand, but in some cases you’ll need to figure it out, as in 1.2.13 (but do that problem without the inhomogeneous term)
• Of course, for this you need to be super comfortable with the Chain Rule/Chen Lu
• Also know how to find the solution of some more general first-order PDEs, like \( u_x + yu_y = 0 \) or \( u_x + 2xy^2 u_y = 0 \) and others (1.2.5)
• Those problems require you to be comfortable with differential equation techniques like solving \( \frac{dy}{dx} = \frac{y}{x} \).
• For all those problems, I might ask you to plug in initial conditions, like 1.2.1
• Solve PDEs by substituting new functions, as in 1.2.2 and 1.2.8 (in that case I would tell you what function to use, like the HW 2 hint)
• Also look at 1.2.9

Note: You can skip sections 1.3, 1.4, and 1.5. You do NOT need to know the derivation of the transport equation I gave in Lecture 4, and you do NOT need to know the types of boundary conditions, and you do NOT need to know the divergence theorem (at least for the midterm). That said, DO check out 1.5.5 and 1.5.6 since they’re good practice with 1.2 and DO know what uniqueness means.

SECTION 1.6: TYPES OF SECOND-ORDER EQUATIONS

• Know the definition of elliptic, hyperbolic, parabolic and figure out if a PDE is elliptic etc. (1.6.1, 1.6.2, 1.6.4, 1.6.6)
• Note that the book and I use slightly different conventions, so decide which one you like more

SECTION 2.1: THE WAVE EQUATION

• Absolutely know how to derive the general solution of the wave equation \( u_{tt} = c^2 u_{xx} \) using the factoring method.

Note: At some point you have to solve \( u_t + cu_x = f(x + ct) \). In that case, for the particular solution, guess \( u(x, t) = aF(x+ct) \) and solve for \( a \) (the book and my methods are slightly too complicated in that respect)
• Do the same thing but with the coordinate method. Here you need to know which coordinates to use (you can guess them because you know the general solution ;)
• Know how to derive D’Alembert’s formula. Again, the book and I give slightly different derivations, so decide which one you like more (2.1.1, 2.1.2)
• You also need to know how to use the factoring method to solve more general PDE like \( u_{xx} - 3u_{xt} - 4u_{tt} = 0 \) (2.1.9, 2.1.10) Note: Just be careful (for instance) that an antiderivative of \((F(4s))^'\) with respect to \( s \) is \( \frac{1}{4}F(4s) \) and not \( F(4s) \).
• Ignore the plucked string example (Example 2 in 2.1), it’s really just for illustrative purposes
• Also check out 2.1.8, it’s fun!

SECTION 2.2: CAUSALITY AND ENERGY

• Ignore domain of dependence/domain of influence, I won’t ask anything about it!
• Know how to use energy methods. This means multiplying your PDE by a function and integrating by parts with respect to x.
  **Note:** I will **NOT** give you the energy here, and you should **NOT** memorize it, but you DO have to know how to derive it. The only thing I will give you is the function you have to multiply it with, like “Multiply the PDE by $u_t$”
• Don’t just expect the wave equation, I might give you a different PDE for which the energy method holds!
• Use the energy method to prove that the only solution of the wave equation with zero initial conditions is the zero function (2.2.1) and know how to derive uniqueness of solutions of the wave equation from that (see Lecture 8)
• Ignore 2.2.2 and 2.2.5

SECTION 2.3: THE DIFFUSION EQUATION

• Know the statement of the Maximum Principle, but you don’t need to know its proof
• Use the maximum principle to show that the only solution of the heat equation with 0 initial and boundary conditions is the zero function (Lecture 12) and know how to derive uniqueness from that (Lecture 12), as well as stability of the solutions (Lecture 12; in that case I would tell you what M is)
• Prove the comparison principle for the heat equation (2.3.6)
• Also check out 2.3.4, it’s a lot of fun :)
• Know how to use energy methods for the heat equation (and possible variations of the heat equation). Again, I would tell you what function to multiply this by. (check out 2.4.15 as well)

SECTION 2.4: DIFFUSION ON THE WHOLE LINE

• You need to know how to derive the fundamental solution $S(x,t)$ in the following sense: Suppose $u(x,t)$ is of the form $\frac{1}{\sqrt{t}} v \left( \frac{x}{\sqrt{t}} \right)$ for some $v = v(y)$ to be found, plug this into $u_t = k u_{xx}$ and solve for $v$. Anything else, including guessing $\frac{x}{\sqrt{t}}$ and solving for $C$ you’re not
responsible for. I highly recommend looking at Lecture 9 instead of the book, since the book’s derivation makes no sense

- You do **NOT** need to memorize the formula for \( S(x, t) \), it will be provided to you if necessary
- Know the definition of convolution and be able to write a solution of \( u_t = ku_{xx} \) with \( u(x, 0) = \phi(x) \) in terms of \( S \) and \( \phi \)
- Know how to show that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \) (see Gaussian Integral) and that \( \int_{-\infty}^{\infty} S(x, t) \, dx = 1 \) (given the formula for \( S \)) (2.4.6, 2.4.7)
- You **DON’T** need to know the definition of Erf (in the book)
- I might ask you to find an explicit formula of \( u \) if \( \phi \) is given, as in Example 2 in the book or as in Lecture 10, but I’ll try to make the algebra more manageable. Although it’s messy, the whole point is just to complete the square with respect to \( y \) (2.4.3)
- You don’t need to understand the part of Lecture 10 with the Dirac Delta functional, and you don’t need to understand the intuition behind convolution (Lecture 10)
- Also check out 2.4.11(a)(b) and 2.4.16-18, those are all good problems.

Finally, you **DON’T** need to know section 2.5 (Comparison of Waves and Diffusions) for the midterm (or the final), it’s a completely optional.