MOCK MIDTERM SOLUTIONS

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PROBLEM 1:

1) $$\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)} = Slope$$

$$\Leftrightarrow \quad \int \cos(y)\,dy = \int sin(x)\,dx$$

$$\Leftrightarrow \quad sin(y) = -\cos(x) + C$$

$$\Leftrightarrow \quad \underbrace{\cos(x) + \sin(y)}_{?} = C$$

2) General solution

u(x,y) = f(?) = f(cos(x)+sin(y))

$$u(x, y) = f \quad \cos(x) + \sin(y))$$

3) Initial Condition

$$\underbrace{u(x,0)}_{(\cos(x))^2} = f(\cos(x) + \sin(\cancel{0}^{0}))$$

$$(\cos(x))^2 = f(\cos(x))$$

Hence: $f(\cos(x)) = (\cos(x))^2$

So: $f(x) = x^2$

4) Conclusion:

$$u(x,y) = f\big(\cos(x) + \sin(y)\big)$$

$$\boxed{u(x,y) = \big(\cos(x) + \sin(y)\big)^2}$$

PROBLEM 2:

1) Note: $x^2+xt-2t^2 = (x+2t)(x-t)$

2) Hence $u_{xx} + u_{xt} - 2u_{tt}$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\,\partial t} - 2\frac{\partial^2}{\partial t^2}\right)u$$

$$= \left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial t}\right)\underbrace{\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u}_{v}$$

$$= \left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial t}\right)v$$

$$= \quad v_x + 2v_t = 0$$

2) But $v_x + 2v_t = 0 \qquad (a = 1, b = 2)$

$\Rightarrow v(x,t) = f(2x - t) \qquad (bx - ay)$

3) Now use $v = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) u$

$$\Rightarrow u_x - u_t = v = f(2x - t)$$

4) To solve this, first solve the homogeneous PDE

$$u_x - u_t = 0 \quad (a = 1, b = -1)$$

=> $u(x,t) = G(x + t)$ (ay - bx; could also have used ax - by)

Now find a particular solution of $u_x - u_t = f(2x - t)$

Guess: $u(x,t) = a\,F(2x-t)$, then:

$$u_x - u_t = f(2x - t)$$

=> $(a\,F(2x-t))_x - (a\,F(2x-t))_t = f(2x-t)$
=> $a\,F'(2x-t)\,(2) - a\,F'(2x-t)\,(-1) = f(2x-t)$
=> $2a\,f(2x-t) + a\,f(2x-t) = f(2x-t)$
=> $3a\,f(2x-t) = f(2x-t)$
=> $3a = 1$
=> $a = 1/3$

Hence a particular solution is u(x,t) = (1/3) F(2x-t) = F(2x-t) (since F is arbitrary)

5) General Solution: $u(x,t) = \underbrace{G(x+t)}_{\text{(Hom. Sol)}} + \underbrace{F(2x-t)}_{\text{(Particular sol)}}$

$$\boxed{u(x,t) = F(2x - t) + G(x+t)} \quad \text{(F, G arbitrary)}$$

PROBLEM 3:

1) $u(x,t) = S(x,t) * \phi(x)$

$$= \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy$$

K = 1/4

$$= \frac{1}{\sqrt{4\pi/\left(\frac{1}{4}\right)t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\left(\frac{1}{4}\right)t}} e^{2y}\,dy$$

$$= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}+2y}\,dy$$

2) Complete the square in the exponent:

$$\frac{-(x-y)^2}{t} + 2y$$

$$= -\frac{1}{t}[y^2 - 2(x+t)y + x^2]$$

$$= -\frac{1}{t}[(\ y-(x+t)\ )^2 - (x+t)^2 + x^2]$$

$$= -\frac{1}{t}[(\ y-(x+t)\ )^2 - x^2 - 2xt - t^2 + x^2]$$

$$= \boxed{\left(-\frac{(\ y-(x+\mathrm{t})\ )^2}{t}\right) + 2x + t}$$

3) Hence

$$u(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(\ y-(x+\mathrm{t})\ )^2}{t}} e^{2x+t}\, dy$$

$$= \frac{e^{2x+t}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left(\frac{y-(x+t)}{\sqrt{t}}\right)^2} dy$$

$$p = \frac{y-(x+\mathrm{t})}{\sqrt{t}}$$

$$\Rightarrow \quad dp = \frac{dy}{\sqrt{t}}$$

$$dy = \sqrt{t}\, dp$$

$$= \frac{e^{2x+t}}{\cancel{\sqrt{\pi}}\cancel{\sqrt{t}}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} \cancel{\sqrt{t}}\, dy}_{\cancel{\sqrt{\pi}}}$$

4) $\boxed{u(x,t) = \mathrm{e}^{2x+t}}$

PROBLEM 4:

1) Let u and v be two solutions and let $w = u - v$

Then $w_{tt} = (u-v)_{tt} = u_{tt} - v_{tt}$
$= c^2 u_{xx} + f - (c^2 v_{xx} + f)$
$= c^2 u_{xx} + f - c^2 v_{xx} - f$
$= c^2 (u_{xx} - v_{xx})$
$= c^2 (u-v)_{xx}$
$= c^2 w_{xx}$

So w solves $w_{tt} = c^2 w_{xx}$

Moreover $w(0,t) = u(0,t) - v(0,t) = g(t) - g(t) = 0$
And similarly for the other boundary/initial conditions

Therefore w solves the following PDE:

$$\begin{cases} w_{tt} = c^2 w_{xx} \\ w(0,t) = 0 \\ w(l,t) = 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

2) Now take this PDE and multiply it by w_t

(this is why I didn't say u_t but a variation of u_t)

$$w_{tt}\, w_t = c^2\, w_{xx}\, w_t$$

And integrate with respect to x from 0 to l

$$\int_0^{\ell} w_{tt}\, w_t\, dx = c^2 \int_0^{\ell} w_{xx}\, w_t\, dx$$

3) Since $w_{tt}\, w_t = 1/2\ d/dt\ (w_t)^2$ the left hand side becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{l} (w_t)^2\,\mathrm{d}x$$

4) For the right-hand-side, integrate by parts with x to get

$$\int_0^{\ell} w_{xx}\, w_t\, dx = w_x(l,t)\, \cancelto{0}{w_t(l,t)} - w_x(0,t)\, \cancelto{0}{w_t(0,t)} - \int_0^{\ell} w_x\, w_{xt}\, dx$$

Note that $w_t(l,t) = d/dt\ (w(l,t)) = d/dt\ (0) = 0$

And similarly $w_t(0,t) = 0$, so the boundary terms are 0

Moreover $w_x\, w_{xt} = d/dt\ (w_x)^2$ so the right hand side ultimately becomes

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{l} (w_x)^2\,\mathrm{d}x$$

5) Finally, equating both sides we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (w_t)^2\,\mathrm{d}x = -c^2\,\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (w_x)^2\,\mathrm{d}x$$

$$\Longrightarrow \quad \frac{d}{dt}\underbrace{\left[\int_0^l (w_t)^2 + c^2 (w_x)^2\,\mathrm{d}x\right]}_{E(t)} = 0$$

=> E'(t) = 0

In other words, the energy is constant, so in particular

E(t) = E(0)

But E(0) = $\displaystyle\int_0^l \left(\cancelto{0}{w_t(x,0)}\right)^2 + \left(\cancelto{0}{w_x(x,0)}\right)^2\,\mathrm{d}x$

(since $w_x(x,0) = (w(x,0))_x = (0)_x = 0$)

So E(0) = 0

And in particular E(t) = $\displaystyle\underbrace{\int_0^l (w_t)^2 + c^2 (w_x)^2\,\mathrm{d}x}$ = E(0) = 0

≥ 0

But this implies $(w_t)^2 + c^2 (w_x)^2 = 0$ and hence $w_t = 0$ and $w_x = 0$

But then this means $w(x,t) = C$ where C is a constant

And in particular $w(x,0) = C$, but $w(x,0) = 0$, so $C = 0$

Hence $w = 0$, so $u - v = 0$, so $\boxed{u = v}$