LECTURE 16: HALF HEAT EQUATION

Thursday, October 31, 2019 2:02 PM

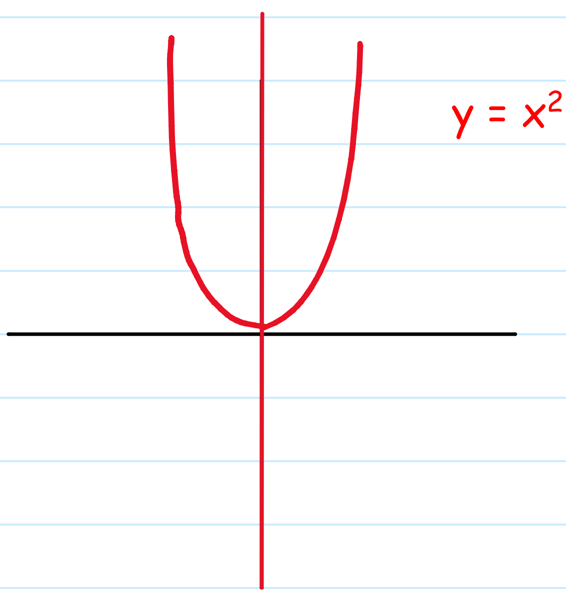
Today: Want to solve the heat equation on the half-line (semi-infinite rod), but first let me "remind" you of a long-forgotten calculus concept:

I- EVEN AND ODD FUNCTIONS

A) DEFINITION

1) f is even if $f(-x) = f(x)$

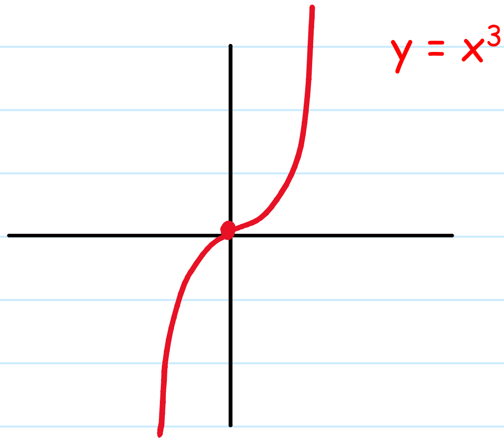
Ex: $f(x) = x^2$



Even functions are symmetric about the y-axis

2) f is odd if $f(-x) = -f(x)$

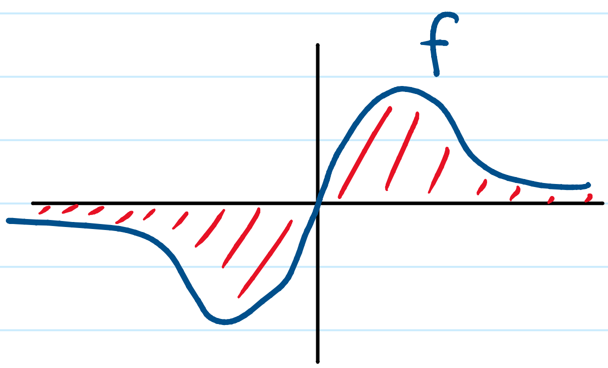
Ex: $f(x) = x^3$



Odd functions are symmetric about the origin

B) IMPORTANT FACTS

1) If f is odd, then $\int_{-\infty}^{\infty} f(x)dx = 0$



2) If f is even, then f' is odd

3) (Even) x (Odd) = Odd

4) If f is odd, then f(0) = 0

C) EVENIFICATION AND ODDIFICATION

Suppose now $f(x)$ is only defined only on $(0,\infty)$

Is there some nice way to extend f to all of $\mathbb{R}$?

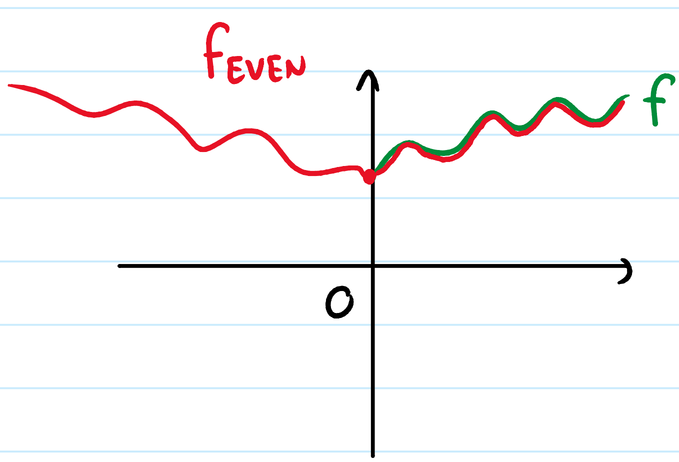
Yes! In fact two of them!

(i) EVENIFICATION (Even extension)

$$\boxed{f_{even}(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x \leq 0 \end{cases}}$$

(f(0) defined by continuity)

Geometrically: You reflect the graph of f about the y-axis



Fact: f_{even} is even, and $f_{even}(x) = f(x)$ if $x \geq 0$
(so in fact it is an even extension of f)

(ii) ODDIFICATION (Odd Extension)

$$\boxed{f_{odd}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x < 0 \end{cases}}$$

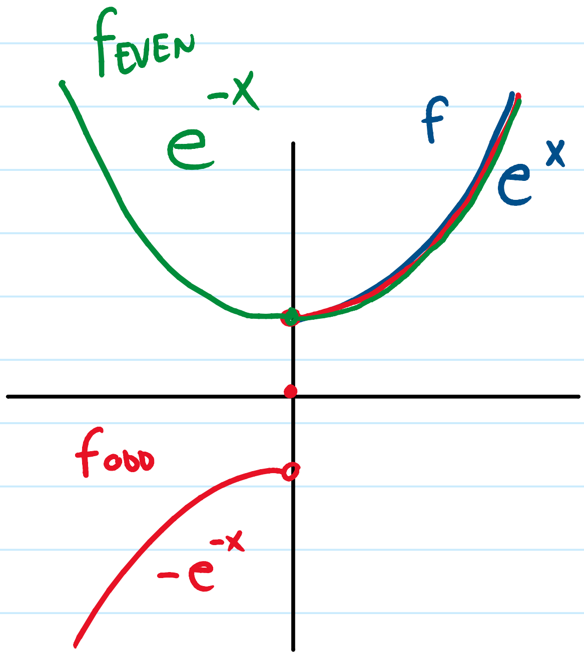
Geometrically: You reflect f about the origin

Fact: f_{odd} is odd and $f_{odd}(x) = f(x)$ if $x > 0$
(so in fact an odd extension)

Example: If $f(x) = e^x$, then:

Then $f_{even}(x) = \begin{cases} e^x \text{ on } [0,\infty) \\ e^{-x} \text{ on } (-\infty, 0] \end{cases}$

$$f_{odd}(x) = \begin{cases} e^x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -e^{-x} \text{ if } x < 0 \end{cases}$$



But what does this have to do with PDE ?????

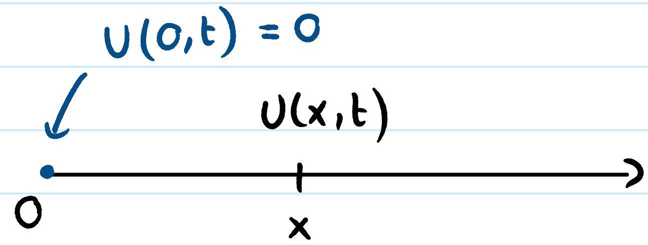
II- HALF HEAT EQUATION

Setting:

Suppose this time you only have a half-infinite rod $x > 0$ and you insulate it on the left to have temperature 0 at all times.

Picture: $u(0,t) = 0$

Picture:



Then you get the following half heat equation (= heat equation on half-line)

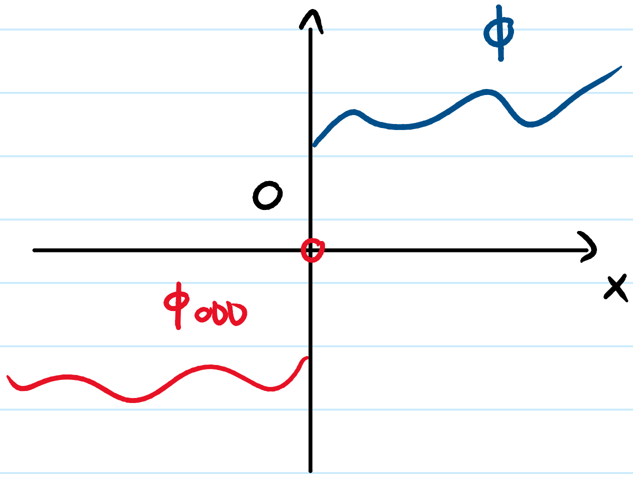
$$\boxed{\begin{cases} u_t = k\, u_{xx} & (x > 0) \\ u(x,0) = \phi(x) & (x > 0;\ \text{Initial condition}) \\ u(0,t) = 0 & (\text{Boundary condition}) \end{cases}}$$

Warning: Here $u(x,t)$ and $\phi(x)$ are only defined for $x > 0$

It would be nice if we could somehow turn this into a problem on the **whole** line $-\infty < x < \infty$ because we know how to solve the latter! And indeed we can by using oddification!

STEP 1: Define

$$\phi_{odd}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \phi(-x) & \text{if } x < 0 \end{cases}$$



And solve

$$\begin{cases} u_t = k\, u_{xx} & (-\infty < x < \infty) \\ u(x,0) = \phi_{odd}(x) \end{cases}$$

(So heat equation on the whole line, but with initial data ϕ_{odd})

Note: Why use oddification? Intuitively, it's because we want $u(0,t) = 0$, and in general if f is odd, we have $f(0) = 0$, so those two might seem related!

STEP 2:

$$\Rightarrow u(x,t) = S(x,t) * \phi_{odd}(x) = \int_{-\infty}^{\infty} S(x-y,t)\, \phi_{odd}(y)\, dy$$

Claim: $u(x,t)$ solves our original problem on the half-line $x > 0$

Why?

1) u still solves $u_t = k\, u_{xx}$ for $x > 0$

2) Moreover, if $x > 0$, then $u(x,0) = \phi_{odd}(x) = \phi(x)$ (by definition of ϕ_{odd} and since $x > 0$ here)

3) Finally

$$u(0,t) = \int_{-\infty}^{\infty} S(0-y,t)\, \phi_{odd}(y)\, dy \qquad (\text{set } x = 0)$$

$$= \int_{-\infty}^{\infty} S(-y,t)\, \phi_{odd}(y)\, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\underbrace{S(y,t)}_{\text{Even}}\, \underbrace{\phi_{odd}(y)}_{\text{Odd}}}_{\text{Odd}}\, dy \quad \left(\text{Since } S(y,t) = \frac{1}{\sqrt{4\pi k t}} e^{\frac{-y^2}{4kt}} \text{ is even in } y\right)$$

$$= \int_{-\infty}^{\infty} \text{ODD}$$

$$= 0$$

So using oddification, we basically get $u(0,t) = 0$ for free!

STEP 4: Explicit formula without ϕ_{odd}

What does this look like explicitly??

Recall $\phi_{odd}(y) = \begin{cases} \phi(y) & \text{if } y > 0 \\ -\phi(-y) & \text{if } y < 0 \end{cases}$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\ \phi_{odd}(y)\, dy$$

$$= \int_{-\infty}^{0} S(x-y,t)\ \phi_{odd}(y)\, dy + \int_{0}^{\infty} S(x-y,t)\ \phi_{odd}(y)\, dy$$

$$= \int_{-\infty}^{0} S(x-y,t)\ (-\phi(-y))\, dy + \int_{0}^{\infty} S(x-y,t)\ \phi(y)\, dy$$

(u-sub: $p = -y$, $dp = -dy \Rightarrow dy = -dp$, $p(-\infty) = \infty$, $p(0) = 0$)

$$= \int_{\infty}^{0} S(x+p,t)\ \phi(p)\, dp + \int_{0}^{\infty} S(x-y,t)\ \phi(y)\, dy$$

$$= -\int_{0}^{\infty} S(x+y,t)\ \phi(y)\, dy + \int_{0}^{\infty} S(x-y,t)\ \phi(y)\, dy$$

(u-sub: $y = p$, and we reversed the order of integration)

STEP 5: Conclusion

$$\boxed{u(x,t) = \int_{0}^{\infty} (S(x-y,t)-S(x+y,t))\ \phi(y)\, dy}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[e^{\frac{-(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y)\, dy$$

(Notice the integral is from 0 to infinity!
Also don't memorize it, but know how to derive it)

Example: Solve

$$\begin{cases} u_t = k\, u_{xx} \quad (x > 0) \\ u(0,t) = 0 \\ u(x,0) = 1 \end{cases}$$

$\phi(x) = 1$

$$u(x,t) = \int_0^{\infty} [S(x-y,t) - S(x+y,t)] \underbrace{1}_{\phi(y)} \, dy$$

$$= \int_0^{\infty} S(y-x,t)\, dy - \int_0^{\infty} S(x+y,t)\, dy$$

(S is even, so $S(x-y,t) = S(-(x-y),t) = S(y-x,t)$)

$$= \int_{-x}^{\infty} S(p,t)\, dp - \int_{x}^{\infty} S(q,t)\, dq$$

($p = y-x$, $dp = dy$, $p(0) = -x$, $p(\infty) = \infty - x = \infty$)
($q = x + y$, $dq = dy$, $q(0) = x$, $q(\infty) = x + \infty = \infty$)

$$= \int_{-x}^{\infty} S(y,t)\, dy + \int_{\infty}^{x} S(y,t)\, dy$$

$$= \int_{-x}^{x} S(y,t)\,dy$$

$$= 2\int_{0}^{x} S(y,t)\,dy \quad \text{(because } S \text{ is even)}$$

$$= \frac{\cancel{2}}{\sqrt{\cancel{4}\pi k t}} \int_0^x \left[e^{\frac{-y^2}{4kt}}\right] dy$$

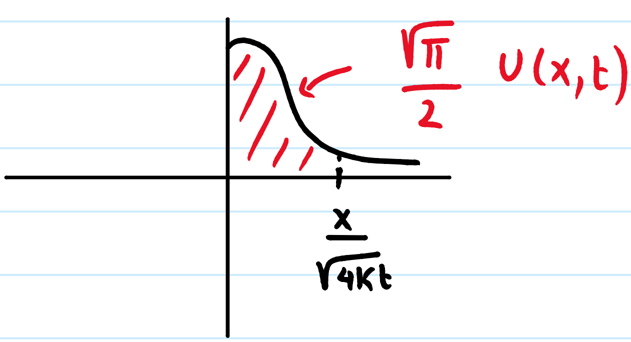
$p = y/(4kt)^{1/2}$

$dp = dy/(4kt)^{1/2}$

$$= \frac{\sqrt{4\cancel{kt}}}{\sqrt{\pi \cancel{kt}}} \int_0^{\frac{x}{\sqrt{4kt}}} \left[e^{-p^2}\right] dp$$

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} \left[e^{-p^2}\right] dp$$

(as explicit as we can get; sometimes called "Error function", u(x,t) is multiple of area under bell curve)



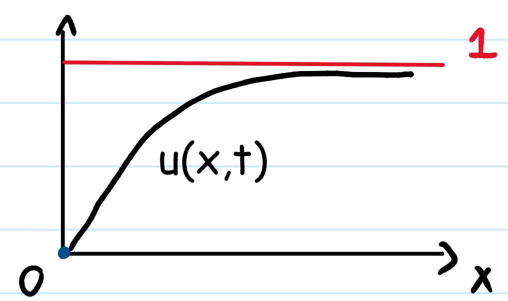
Interpretation: Notice:

$$u(0,t) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{0}{\sqrt{4kt}}} \left[e^{-p^2} \right] dp = 0$$

$$u(\infty, t) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \left[e^{-p^2} \right] dp = 1$$

So if $t > 0$ is fixed, $u(x,t)$ increases from 0 (where it's insulated) to 1:

Picture: t fixed



As t increases, we still have the same phenomenon, but The graph of u gets stretched horizontally