LECTURE 17: REFLECTION OF WAVES

Thursday, October 31, 2019 10:27 PM

Let's continue our half-line extravaganza!

I- NEUMANN PROBLEM

What if we now want to solve

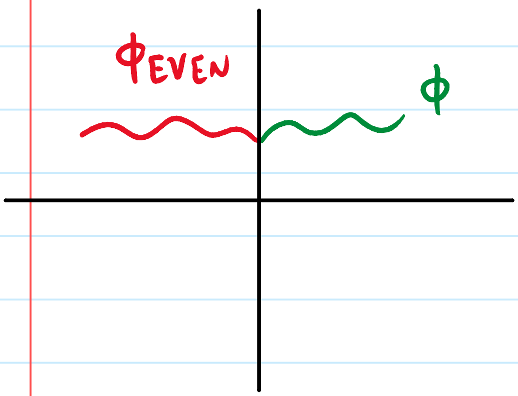
$$\begin{cases} u_t = k\, u_{xx} & (x > 0) \\ u_x(0,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

This time we want the *derivative* of a function to be odd, so we want the function to be *even*

STEP 1: Evenify f

Recall:

$$\phi_{even}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ \phi(-x) & \text{if } x < 0 \end{cases}$$



STEP 2:

Solve

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \phi_{even}(x) \end{cases}$$

$$\Rightarrow u(x,t) = S(x,t) * \phi_{even}(x)$$

And just like last time you eventually get

$$\boxed{u(x,t) = \int_0^{\infty} [S(x-y,t) + S(x+y,t)]\ \phi(y)\, dy}$$

(see Homework 5 for details)

Example: Solve

$$\begin{cases} u_t = k\, u_{xx} & (x > 0) \\ u_x(x,0) = 0 \\ u(0,t) = 1 \end{cases}$$

By above, with $\phi(y) = 1$, get

$$u(x,t) = \int_0^{\infty} [S(x-y,t) + S(x+y,t)]\ 1\, dy$$

$$= \int_x^{-\infty} S(p,t)\,(-dp) + \int_x^{\infty} S(q,t)\, dy$$

$(p = x - y,\ q = x + y)$

$$= \int_{-\infty}^{x} S(y,t)\,dy + \int_{x}^{\infty} S(y,t)\,dy$$

$$= \int_{-\infty}^{\infty} S(y,t)\,dy$$

$= 1$ (by definition of S)

II- REFLECTION OF WAVES

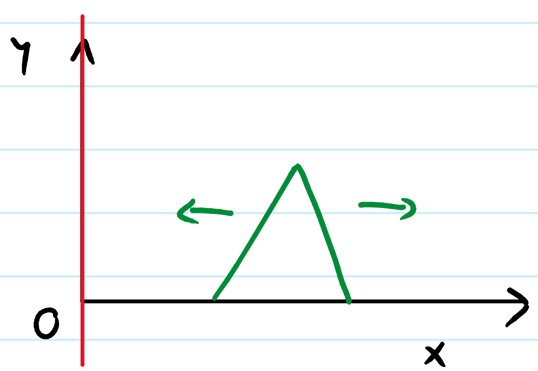
The really neat thing is that the exact same method also works for the wave equation!

Setting: Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0) \\ u(0,t) = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Wave equation, but this time you have a wall/barrier at $x = 0$

Picture:



Remember that for the wave equation, the initial position $\phi(x)$ splits into two parts, one going to the right and the other going to the left. The right part won't be affected, but the left part is eventually going to hit the wall and change direction!

Picture:

Let's see if this indeed happens!

STEP 1: Oddify both ϕ and ψ

$$\phi_{odd}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ -\phi(-x) & \text{if } x < 0 \end{cases}$$

$$\psi_{odd}(x) = \begin{cases} \psi(x) & \text{if } x > 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}$$

STEP 2: Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi_{odd}(x) \\ u_t(x,0) = \psi_{odd}(x) \end{cases}$$

=> D'Alembert:

$$u(x,t) = 1/2\ (\phi_{odd}(x-ct) + \phi_{odd}(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi_{odd}(s)\, ds$$

Then, as before, u solves our original problem!

Ex: $u(0,t) = 1/2\ (\phi_{odd}(-ct) + \phi_{odd}(ct)) + 1/(2c) \int_{-ct}^{ct} \underbrace{\psi_{odd}(s)}_{\text{ODD}}\, ds$

$$= 1/2\ (\cancel{-f_{odd}(ct)} + \cancel{f_{odd}(ct)}) + 0$$

$$= 0$$

STEP 3: Write in terms of ϕ and ψ

Note: We always have $x + ct > 0$ (since $x > 0$), so in D'Alembert above we just need to argue in terms of the sign of $x-ct$

CASE 1: $x-ct > 0$ (=> $t < x/c$ think "t small")

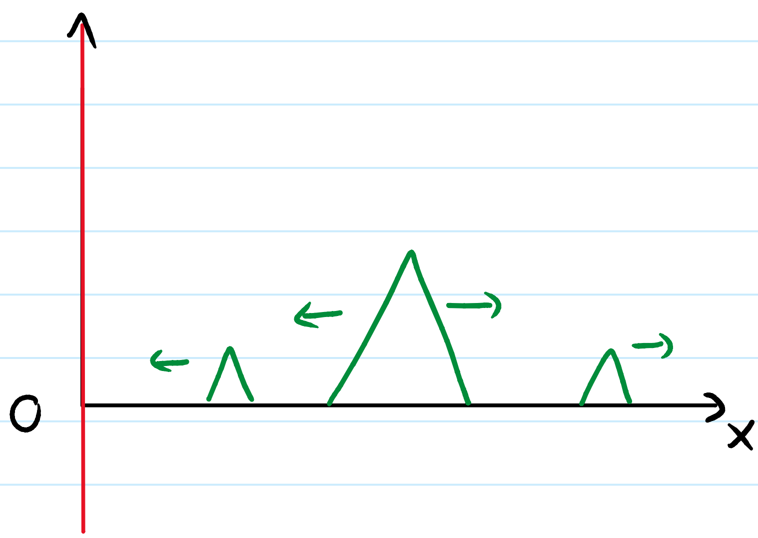
Then $\phi_{odd}(x-ct) = \phi(x-ct)$ (and $\phi_{odd}(x+ct) = \phi(x+ct)$) and $\psi_{odd}(s) = \psi(s)$ on $[x-ct, x+ct]$ (since $x-ct > 0$), so

$$\boxed{u(x,t) = 1/2\,(\phi(x-ct) + \phi(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds}$$

(usual d'Alembert's formula)

Interpretation: Before you hit the wall, the wave just goes on as usual

Picture: t small



CASE 2: $x-ct < 0$ ($\Rightarrow t > x/c$ think "t large")

Then $\phi_{odd}(x-ct) = -\phi(-(x-ct)) = -\phi(ct-x)$
(and $\phi_{odd}(x+ct) = \phi(x+ct) = \phi(ct+x)$)

And

$$\int_{x-ct}^{x+ct} \psi_{odd}(s)\, ds = \int_{x-ct}^{0} \psi_{odd}(s)\, ds + \int_{0}^{x+ct} \psi_{odd}(s)\, ds$$

$$\int_{x-ct} \psi_{odd}(s)\, ds = \int_{x-ct} \psi_{odd}(s)\, ds + \int_{0} \psi_{odd}(s)\, ds$$

$$= \int_{x-ct}^{0} -\psi(-s)\, ds + \int_{0}^{x+ct} \psi(s)\, ds$$

$p = -s$

$dp = -ds$

$$= \int_{-x+ct}^{0} \psi(p)\, dp + \int_{0}^{x+ct} \psi(s)\, ds$$

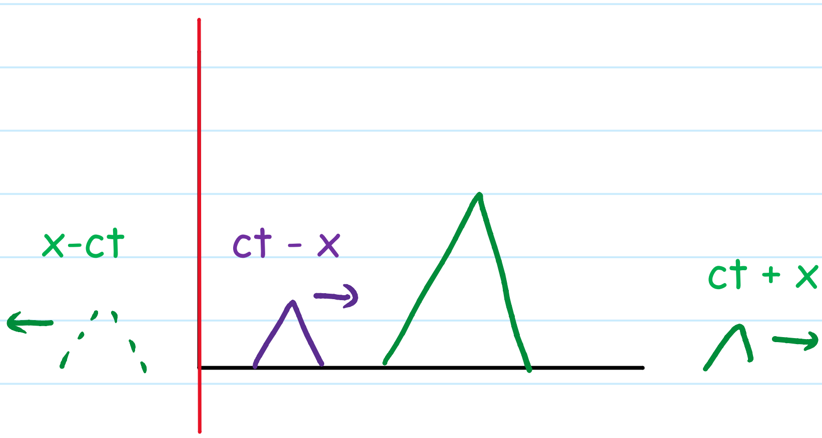
$$= \int_{ct-x}^{ct+x} \psi(s)\, ds$$

And therefore

$$\boxed{u(x,t) = 1/2 \; (\phi(ct+x) - \phi(ct-x)) + 1/(2c) \int_{ct-x}^{ct+x} \psi(s)\, ds}$$

"Reflected D'Alembert's Formula"

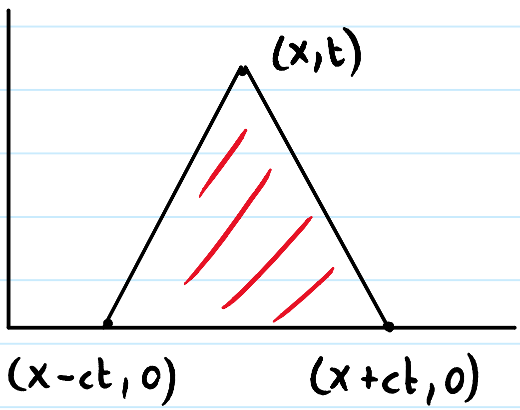
Picture: t large



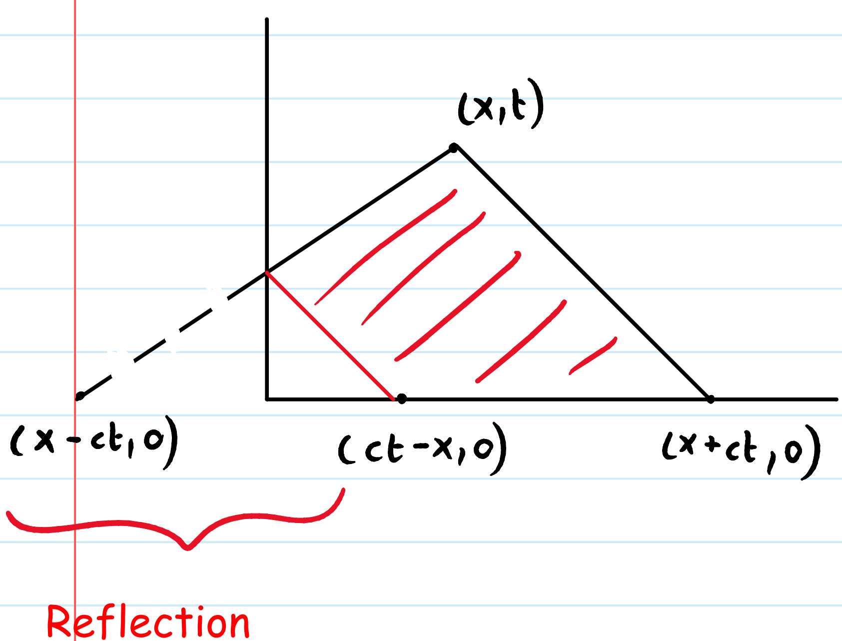
Note: Can also explain this with the

Domain of dependence
(= given (x,t), which initial values does u(x,t) depend on?)

Case 1: t small



Case 2: t large

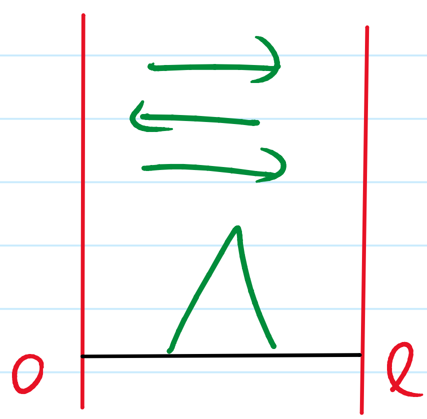


Remarks:

1) The case $u_x(0,t) = 0$ is similar (this time you evenify ϕ and ψ, see Homework 5)

2) In theory, can also treat the problem where there are two walls:

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad (0 < x < l) \\ u(0,t) = 0,\ u(l,t) = 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \end{cases}$$



This time there will be many reflections back and forth, and the formula gets quickly out of hand (but feel free to check out 3.2 if you're interested)

III- INHOMOGENEOUS PROBLEM

Lastly, What if you don't require $u = 0$ at the endpoint, but instead $u = 7$?

This actually can be dealt with with one simple trick!

Example: (going back to heat equation)

$$\begin{cases} u_t = k\, u_{xx} & (x > 0) \\ u(x,0) = \phi(x) & (x > 0) \\ u(0,t) = 7 \end{cases}$$

Trick: Let $\boxed{v(x,t) = u(x,t) - 7}$

Then $v_t = k\, v_{xx}$ (check)

$v(x,0) = u(x,0) - 7 = \phi(x) - 7$

$v(0,t) = u(0,t) - 7 = 7 - 7 = 0$

So v solves:

$$\begin{cases} v_t = k\, v_{xx} & (x > 0) \\ v(x,0) = \phi(x) - 7 \\ v(0,t) = 0 \end{cases}$$

So by the half heat formula with $\phi(x) - 7$ instead of $\phi(x)$

$$v(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)](\phi(y) - 7)\, dy$$

$$u(x,t) - 7 = \int_0^\infty [S(x-y,t) - S(x+y,t)](\phi(y) - 7)\, dy$$

$$u(x,t) = 7 + \int_0^\infty [S(x-y,t) - S(x+y,t)](\phi(y) - 7)\, dy$$

Note: Same trick works with wave equation. In that case $\phi(x)$ becomes $\phi(x) - 7$, but $\psi(x)$ stays the same!

Example:

$$\begin{cases} u_t = k\, u_{xx} & (x > 0) \\ u(x,0) = \phi(x) & (x > 0) \\ u_x(0,t) = 7 & \end{cases}$$

This time $v(x,t) = u(x,t) - 7x$
(basically want a function whose x derivative is 7)

Then same process, but $\phi(x)$ becomes $\phi(x) - 7x$ and you get

$$u(x,t) = 7x + \int_0^\infty [S(x-y,t) + S(x+y,t)]\, (\phi(y) - 7y)\, dy$$

(And again, same thing with the wave equation, $\phi(x)$ becomes $\phi(x) - 7x$, but $\psi(x)$ is unchanged)

Note: Finally, can also solve truly inhomogeneous versions of heat and wave equations (on the whole line and the half line), such as $u_t = k\, u_{xx} + f(x,t)$ (using the usual homogeneous + particular solution trick), and you can check out 3.3 and 3.4 if you like.