LECTURE 18: SEPARATION OF VARIABLES (I)

Monday, November 4, 2019 12:47 PM

Welcome to the most important PDE technique of the course: Separation of variables! It's **the** technique that will help us solve all the remaining PDEs (like the rod of finite length, and even Laplace's equation)

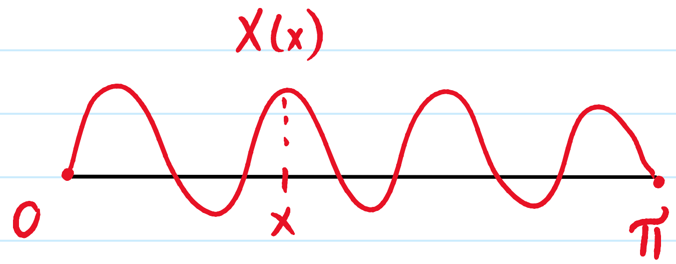
LONG technique, but will do this at least 4 times!

I- BOUNDARY VALUE PROBLEM

Before we do that, let's solve a problem that is purely ODE, but will be *crucial* in our process.

Example: Find a nonzero solution $X(x)$ of

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$



Idea: The solutions of, for example, $X'' = -X$ (which involve cos & sin) are very different from the solutions of $X'' = X$ (which involve exponential terms), so we need to argue in terms of the sign of λ

CASE 1: $\lambda > 0$

Then $\lambda = \omega^2$ for some $\omega > 0$

(**Ex:** If $\lambda = 4$, then $\lambda = 4 = 2^2$ so $\omega = 2$)

Why: If you don't assume that, you'll get nasty $\sqrt{\lambda}$ everywhere

Then:

$X'' = \lambda X \Rightarrow X'' = \omega^2 X \Rightarrow X'' - \omega^2 X = 0$

From Math 3D (ODE), we get:

<u>Auxiliary equation</u>: $r^2 - \omega^2 = 0 \Rightarrow r^2 = \omega^2 \Rightarrow r = \pm\omega$

$\Rightarrow X(x) = A e^{\omega x} + B e^{-\omega x}$

But $X(0) = A e^{\omega 0} + B e^{-\omega 0} = A + B = 0$ (since $X(0) = 0$)

$\Rightarrow B = -A$

So $X(x) = A e^{\omega x} - A e^{-\omega x}$

But $X(\pi) = 0 \Rightarrow A e^{\omega\pi} - A e^{-\omega\pi} = 0$

$\Rightarrow \cancel{A}(e^{\omega\pi} - e^{-\omega\pi}) = 0$

$\Rightarrow e^{\omega\pi} - e^{-\omega\pi} = 0$

($A \neq 0$, because otherwise $X(x) = A e^{\omega x} - A e^{-\omega x} = 0$)

$\Rightarrow e^{\omega\pi} = e^{-\omega\pi}$

$\Rightarrow \omega\pi = -\omega\pi$

$\Rightarrow \omega = 0$

But then $\lambda = \omega^2 = 0$ which is not positive! =><=

CASE 2: $\lambda = 0$

Then $X'' = 0\,X \Rightarrow X''(x) = 0$
$\Rightarrow X(x) = Ax + B$

$X(0) = A0 + B = B = 0$, so $X(x) = Ax$

$X(\pi) = A\pi = 0 \Rightarrow A = 0$, but then $X(x) = 0x = 0$ =><=

CASE 3: $\lambda < 0$

Then $\lambda = -\omega^2$ for some $w > 0$

(**Ex:** $\lambda = -4 = -2^2$)

$X'' = \lambda X \Rightarrow X'' = -\omega^2 X \Rightarrow X'' + \omega^2 X = 0$

Aux: $r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm\, \omega i$

$X(x) = A\cos(\omega x) + B\sin(\omega x)$

$X(0) = A\cos(\omega 0) + B\sin(\omega 0)$
$= A\,1 + B\,0$
$= A = 0$ (since $X(0) = 0$)

So $X(x) = 0\cos(\omega x) + B\sin(\omega x) = B\sin(\omega x)$

$X(\pi) = \cancel{B}\sin(\omega\pi) = 0$

$\Rightarrow \sin(\omega\pi) = 0$

$\Rightarrow \omega\pi = \pi m \ (m = 1, 2, \ldots)$

$\Rightarrow \omega = m$

Answer: For every m, we have a solution,

$X(x) = \sin(\omega x) = \sin(mx) \quad (m = 1, 2, \ldots)$

Note: This problem is sometimes called an eigenvalue problem.

The eigenvalues of $X'' = \lambda X$ are $\lambda = -\omega^2 = -m^2$

And the eigenfunctions are $X(x) = \sin(mx)$

(Compare to $Ax = \lambda x$, from linear algebra, here have infinitely many eigenvalues and eigenvectors)

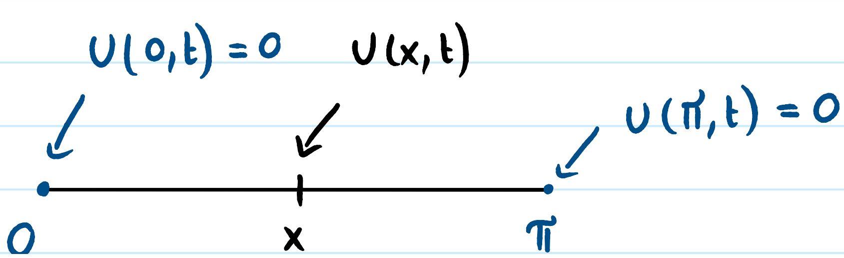
II- SETTING

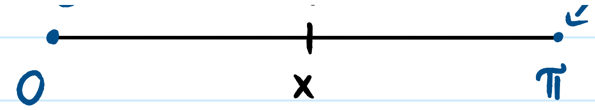
Now let's go back to PDEs and see why we need the problem above

Example: Solve

$$\begin{cases} u_t = k\, u_{xx} \quad (0 < x < \pi) \\ u(0,t) = 0,\ u(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Picture:





It's the usual finite rod problem that's insulated at the endpoints, but this time we actually want to find a solution.

Remarks:

1) We already showed uniqueness using energy methods/maximum principle, so everything just boils down to finding *A* solution.
2) We want to find NONZERO solutions (the zero solution isn't very interesting)

Motivation: For the heat equation on the whole line, we assumed u had a special form

$$u(x,t) = 1/\sqrt{t}\, v(x/\sqrt{t})$$

And we then turned the heat equation into an ODE for v:

$$v'' + (y/2)\, v' + (1/2)\, v = 0 \quad \text{<- MUUUCH simpler!}$$

Here we want to do something similar.

III- SEPARATION OF VARIABLES

STEP 1: Separation of variables

1) Assume u(x,t) has a special form, namely:

$$\boxed{u(x,t) = X(x)\, T(t)} \qquad (*)$$

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Where X is a function of x and T is a function of t only

Ex: $\sin(x)\cos(t)$ is of this form, and so is $(x^2 + 1)(t^3 - 4)$, but $\ln(3x + 2t)$ is not of this form!

2) Plug $(*)$ into $u_t = k\, u_{xx}$

$$(X(x)\,T(t))_t = k\,(X(x)\,T(t))_{xx}$$

$$X(x)\,(T(t))_t = k\,(X(x))_{xx}\,T(t)$$

$$X(x)\,T'(t) = k\,X''(x)\,T(t)$$

3) Put all the X terms on one side and all the T terms on the other side

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)}$$

Always put any extra terms like k (or c^2) with T
(because we want to keep the X equation as simple as possible)

STEP 2: Constant

1) Look at the quantity $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{k\,T(t)}$

On the one hand,

$$\left(\frac{X''(x)}{X(x)}\right)_t = 0$$

But also

$$\left(\frac{X''(x)}{X(x)}\right)_x = \left(\frac{T'(t)}{kT(t)}\right)_x = 0$$

So the $\frac{X''(x)}{X(x)}$ doesn't depend on x or t, so it is constant!

Conclusion: $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{constant} = \lambda$

$$\Rightarrow \frac{X''(x)}{X(x)} = \lambda \Rightarrow X''(x) = \lambda X(x)$$

And $\frac{T'(t)}{kT(t)} = \lambda \Rightarrow T'(t) = k\lambda\, T(t)$

GOOD! We're on the right track; instead of a PDE, we get 2 ODEs.

2) Focus on X(x) equation! (don't touch T until the end!!!)

So far: $X''(x) = \lambda X(x)$

Now use the boundary conditions:

$u(0,t) = 0 \Rightarrow X(0)\,\cancel{T(t)} = 0$ (since $u(x,t) = X(x)T(t)$)

$\Rightarrow \underline{X(0) = 0}$

(Why can you cancel out T? If $T = 0$, then $u = XT = 0$, which wouldn't be interesting. So pick a t where $T(t)$ is nonzero, and cancel out T there)

Similarly $u(\pi,t) = 0 \Rightarrow X(\pi)\cancel{T(t)} = 0 \Rightarrow \underline{X(\pi) = 0}$

Hence we get the ODE

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

Same ODE as the beginning of lecture!!!

STEP 3: Boundary-value problem

INSERT PREVIOUS EXAMPLE

Conclusion: $\lambda = -m^2$ $(m = 1, 2, \ldots)$

$X(x) = \sin(mx)$ $(m = 1, 2, \ldots)$

STEP 4: T equation

$$\frac{T'}{kT} = \lambda = -m^2$$

$\Rightarrow T' = -m^2 k T$

$\Rightarrow T(t) = e^{-m^2 kt}$ (ignore constant C for now)

Conclusion: For every $m = 1, 2, \ldots$

$u(x,t) = X(x)T(t) = \sin(mx)\, e^{-m^2 kt}$ is a solution of our PDE

STEP 5: Linearity

Important observation: Since the PDE is **linear**, any linear combo of the above solution is also a solution!

So

$$u(x,t) = A_1 \sin(1x)\, e^{-(1)^2 kt} + A_2 \sin(2x)\, e^{-(2)^2 kt} + \ldots$$

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx)\, e^{-m^2 kt}}$$ solves the PDE

STEP 6: Initial Condition

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx)\, e^{-m^2 k0} = \sum_{m=1}^{\infty} A_m \sin(mx)$$

$$\boxed{x^2 = \sum_{m=1}^{\infty} A_m \sin(mx)}$$

WTF

=> BIG QUESTION:

Can you find A_m such that the above is true?
In other words, can you write x^2 as a linear combo of sines?

YES, see Chapter 5

(In fact, this is precisely why Fourier series were invented)

Picture:

IV- EASIER PROBLEM

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = 0,\ u(\pi,t) = 0 \\ u(x,0) = 3\sin(2x) + 4\sin(3x) \end{cases}$$

Steps 1-5 still work:

$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx)\, e^{-m^2 kt}$$

This time:

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) \quad = 3\sin(2x) + 4\sin(3x)$$

$$\Rightarrow \quad A_1 \sin(x) + A_2 \sin(2x) + A_3 \sin(3x) + \dots$$
$$= 0\sin(x) + 3\sin(2x) + 4\sin(3x)$$

$A_1 = 0$, $A_2 = 3$, $A_3 = 4$, $A_4 = 0$, etc.

(just compare the coefficients)

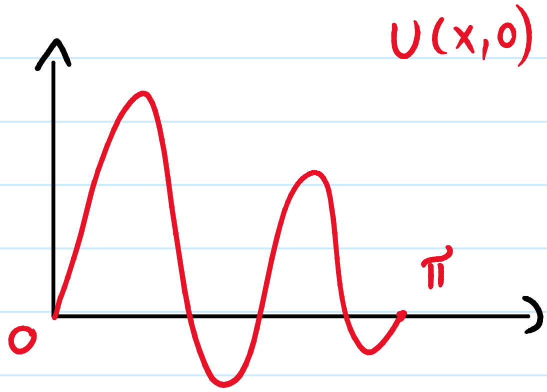
$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx)\, e^{-m^2 kt}$$

$$= \underbrace{3}_{A_2} \sin(2x)\, e^{-2^2 kt} + \underbrace{4}_{A_3} \sin(3x)\, e^{-3^2 kt}$$

$$\boxed{u(x,t) = 3\sin(2x)\, e^{-4kt} + 4\sin(3x)\, e^{-9kt}}$$

Interpretation: Solution starts as $3\sin(2x) + 4\sin(3x)$, but eventually dies down because the exponential terms go to 0

Picture:



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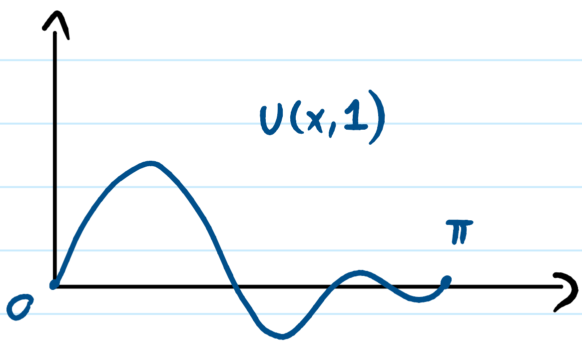

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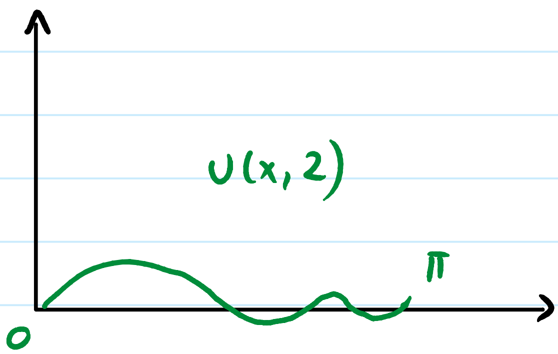

$\Rightarrow$