LECTURE 19: SEPARATION OF VARIABLES (II)

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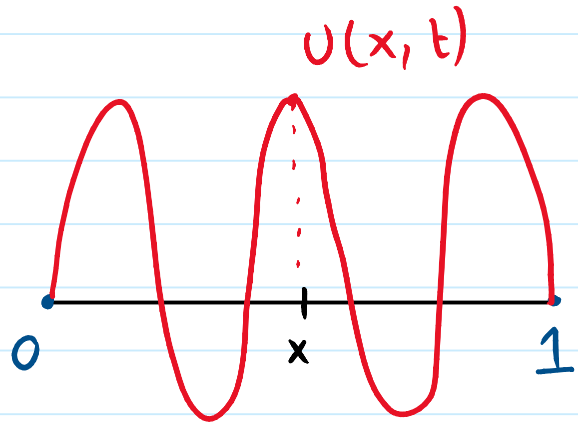
This time we'll separate variables, but with the wave equation!

I- SETTING

Example: This time solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad (0 < x < 1) \\ u(0,t) = 0, \ u(1,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Picture:



II- SEPARATION OF VARIABLES

STEP 1: Separation of variables

1) Suppose:

$$\boxed{u(x,t) = X(x)\, T(t)} \qquad (*)$$

2) Plug (*) into $u_{tt} = c^2 u_{xx}$

$$(X(x)\,T(t))_{tt} = c^2\,(X(x)\,T(t))_{xx}$$

$$X(x)\,T''(t) = c^2\,X''(x)\,T(t)$$

3) Again, put all the T terms on one side, and all the X terms on the other side, making sure the constants go with T

$$\frac{T''(t)}{c^2\,T(t)} = \frac{X''(x)}{X(x)}$$

Like last time, this implies that everything is constant (because the left-hand-side only depends on t whereas the right-hand-side only depends on x)

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2\,T(t)} = \lambda$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \lambda \Rightarrow X''(x) = \lambda\,X(x)$$

And $\dfrac{T''(t)}{c^2 T(t)} = \lambda \Rightarrow T''(t) = c^2 \lambda\,T(t)$

STEP 2: X(x) equation

So far: $X''(x) = \lambda\,X(x)$

Now use the boundary conditions:

$u(0,t) = 0 \Rightarrow \cancel{X(0)}\, T(t) = 0 \Rightarrow \underline{X(0) = 0}$

(Again, can cancel out because otherwise get 0 solution)

Similarly $u(1,t) = 0 \Rightarrow \cancel{X(1)}\, T(t) = 0 \Rightarrow \underline{X(1) = 0}$

Hence we get the ODE

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

STEP 3: Boundary-value problem

Again, argue in terms of the sign of λ

CASE 1: $\lambda > 0$

Then $\lambda = \omega^2$ for some $\omega > 0$

Then:

$X'' = \lambda X \Rightarrow X'' = \omega^2 X \Rightarrow X'' - \omega^2 X = 0$

<u>Aux</u>: $r^2 - \omega^2 = 0 \Rightarrow r^2 = \omega^2 \Rightarrow r = \pm\omega$

$\Rightarrow X(x) = A e^{\omega x} + B e^{-\omega x}$

But $X(0) = A e^{\omega 0} + B e^{-\omega 0} = A + B = 0$ (since $X(0) = 0$)

$\Rightarrow B = -A$

So $X(x) = A\, e^{\omega x} - A\, e^{-\omega x}$

But $X(1) = 0 \Rightarrow A\, e^{\omega 1} - A\, e^{-\omega 1} = 0$

$$\Rightarrow \cancel{A}(e^{\omega} - e^{-\omega}) = 0$$
$$\Rightarrow e^{\omega} - e^{-\omega} = 0$$
$$\Rightarrow e^{\omega} = e^{-\omega}$$
$$\Rightarrow \omega = -\omega$$
$$\Rightarrow \omega = 0$$

But then $\lambda = \omega^2 = 0 \Rightarrow\Leftarrow$ (since we assumed $\lambda > 0$)

CASE 2: $\lambda = 0$

Then $X'' = 0\, X \Rightarrow X''(x) = 0$

$$\Rightarrow X(x) = Ax + B$$

$X(0) = A0 + B = B = 0$, so $X(x) = Ax$

$X(1) = A1 = A = 0 \Rightarrow A = 0$, but then $X(x) = 0x = 0 \Rightarrow\Leftarrow$

CASE 3: $\lambda < 0$

Then $\lambda = -\omega^2$ for some $w > 0$

$X'' = \lambda\, X \Rightarrow X'' = -\omega^2\, X \Rightarrow X'' + \omega^2\, X = 0$

Aux: $r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm\, \omega i$

$X(x) = A\cos(\omega x) + B\sin(\omega x)$

$X(0) = A\cos(\omega 0) + B\sin(\omega 0)$
$= A\,1 + B\,0$
$= A = 0$ (since $X(0) = 0$)

So $X(x) = 0\cos(\omega x) + B\sin(\omega x) = B\sin(\omega x)$

$X(1) = \cancel{B}\sin(\omega) = 0$
$\Rightarrow \sin(\omega) = 0$
$\Rightarrow \omega = \pi m$ $(m = 1, 2, \ldots)$

Answer: For every m, we have a solution,

$X(x) = \sin(\omega x) = \sin(\pi m x)$ $(m = 1, 2, \ldots)$

Conclusion: $\lambda = -(\pi m)^2$ $(m = 1, 2, \ldots)$

$X(x) = \sin(\pi m x)$ $(m = 1, 2, \ldots)$

Note: Last time we had $\lambda = -m^2$ and $X(x) = \sin(mx)$, but that's because we worked on the interval $(0,\pi)$

STEP 4: T equation

$$\frac{T''}{c^2 T} = \lambda = -(\pi m)^2$$

$\Rightarrow T'' = -c^2 (\pi m)^2 T$

$\Rightarrow T'' = -(\pi m c)^2 T$ $\qquad (r^2 = -(\pi m c)^2 \Rightarrow r = +/- \pi m c i)$

=> $T(t) = A\cos(\pi mct) + B\sin(\pi mct)$

Conclusion: For every $m = 1, 2, \ldots$

$u(x,t) = X(x)T(t) = (A\cos(\pi mct) + B\sin(\pi mct))\sin(\pi mx)$
is a solution of our PDE

STEP 5: Linearity

Take linear combos (= sum over m and replace A and B by A_m and B_m to emphasize that your constants depend on m)

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} [A_m\cos(\pi mct) + B_m\sin(\pi mct)]\sin(\pi mx)}$$

STEP 6: Initial Condition

$$u(x,0) = \sum_{m=1}^{\infty} [A_m \underbrace{\cos(\pi mc0)}_{1} + B_m \cancelto{0}{\sin(\pi mc0)}]\sin(\pi mx)$$

$$x^2 = \sum_{m=1}^{\infty} A_m\sin(\pi mx)$$

Same problem as last time!

WTF

$$\boxed{x^2 = \sum_{m=1}^{\infty} A_m \sin(\pi m x)}$$

=> BIG QUESTION:

Can you find A_m such that the above expansion is true?
In other words, can you write x^2 as a linear combo of sines?

YES, see Chapter 5

(In fact, this is precisely why Fourier series were invented)

Picture:

STEP 7: Initial velocity

$$u(x,t) = \sum_{m=1}^{\infty} \left[A_m \cos(\pi m c t) + B_m \sin(\pi m c t)\right] \sin(\pi m x)$$

$$u_t(x,t) = \sum_{m=1}^{\infty} \left([A_m \cos(\pi mct) + B_m \sin(\pi mct)] \sin(\pi mx) \right)_t$$

$$= \sum_{m=1}^{\infty} [(-\pi mcA_m) \sin(\pi mct) + (\pi mc\, B_m) \cos(\pi mct)] \sin(\pi mx)$$

$(t = 0)$

$$u_t(x,0) = \sum_{m=1}^{\infty} [(-\pi mcA_m) \underbrace{\sin(\pi mc0)}_{0} + (\pi mc\, B_m) \underbrace{\cos(\pi mc0)}_{1}] \sin(\pi mx)$$

$$e^x = \sum_{m=1}^{\infty} (\pi mc\, B_m) \sin(\pi mx)$$

$$e^x = \sum_{m=1}^{\infty} \tilde{B}_m \sin(\pi mx)$$

$(\tilde{B}_m = \pi mc\, B_m)$

SAME QUESTION!!! Can you write e^x as a sum of sines?

(So it seems like a pretty BIG deal to do that!)

So right now, we've reached an impasse, which we'll overcome in Chapter 5.

III- EASIER PROBLEM

Example: Same problem, but

$u(x,0) = \sin(2\pi x) + 3\sin(3\pi x)$

$u_t(x,0) = 4\sin(2\pi x)$

Everything we've shown so far is still true:

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} [A_m \cos(\pi mct) + B_m \sin(\pi mct)] \sin(\pi mx)}$$

But this time we can solve for the constants:

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(\pi mx)$$

$$= A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \dots$$

$$= 0 \sin(\pi x) + 1 \sin(2\pi x) + 3\sin(3\pi x)$$

$\Rightarrow A_1 = 0,\ A_2 = 1,\ A_3 = 3$, all other $A_m = 0$

$$u_t(x,0) = \sum_{m=1}^{\infty} (\pi mcB_m) \sin(\pi mx)$$

$$= \pi c\, B_1 \sin(\pi x) + 2\pi c\, B_2 \sin(2\pi x) + 3\pi c\, B_3 \sin(3\pi x) + \dots$$

$$= 0 \sin(\pi x) + 4\sin(2\pi x) + 0\sin(3\pi x) + \dots$$

$\pi c\, B_1 = 0 \Rightarrow B_1 = 0$

$2\pi c\, B_2 = 4 \Rightarrow B_2 = 4/(2\pi c) = 2/(\pi c)$

$\pi m c B_m = 0 \Rightarrow B_m = 0$ (for $m = 3, 4, \ldots$)

Solution: (notice: No more sums because most terms are 0)

$$u(x,t) = \sum_{m=1}^{\infty} [A_m \cos(\pi m c t) + B_m \sin(\pi m c t)] \sin(\pi m x)$$

$$[\underbrace{1}_{A_2} \cos(2\pi c t) + \underbrace{2/(\pi c)}_{B_2} \sin(2\pi c t)] \sin(2\pi x) + [\underbrace{3}_{A_3} \cos(3\pi c t)] \sin(3\pi x)$$

Example: Same but $u(x,0) = \sin(2\pi x)$, $u_t(x,0) = 0$

Can show $u(x,t) = \cos(2\pi c t)\sin(2\pi x)$

Interpretation: Solution starts as $\sin(2\pi x)$ and then oscillates back and forth (just like we had for D'Alembert's formula)

Picture: (Here $c = 1$)