LECTURE 20: SEPARATION OF VARIABLES (III)

Monday, November 4, 2019 12:47 PM

This time we'll separate variables, but with a Neumann condition

I- SETTING

Example: This time solve

$$\begin{cases} u_t = k\, u_{xx} \quad (0 < x < \pi) \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0 \\ u(x,0) = x \end{cases}$$

(Heat equation, but this time the velocity at the endpoints is 0)

II- SEPARATION OF VARIABLES

STEP 1: Separation of variables

1) Suppose:

$$\boxed{u(x,t) = X(x)\, T(t)} \qquad (*)$$

2) Plug $(*)$ into $u_{tt} = c^2 u_{xx}$

$$(X(x)\, T(t))_t = k\, (X(x)\, T(t))_{xx}$$

$$X(x)\, T'(t) = k\, X''(x)\, T(t)$$

3) Again, put all the T terms on one side, and all the X terms on the other side, making sure the constants go with T

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)}$$

Like last time, this implies that everything is constant (because the left-hand-side only depends on t whereas the right-hand-side only depends on x)

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{k\,T(t)} = \lambda$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \lambda \Rightarrow X''(x) = \lambda\, X(x)$$

And $\dfrac{T'(t)}{kT(t)} = \lambda \Rightarrow T'(t) = k\,\lambda\, T(t)$

STEP 2: X(x) equation

So far: $X''(x) = \lambda\, X(x)$

NEW BOUNDARY CONDITIONS

$u(x,t) = X(x)\, T(t)$

$u_x(x,t) = (X(x)\, T(t))_x = X'(x)\, T(t)$

$u_x(0,t) = X'(0)\, T(t) = 0 \Rightarrow \underline{X'(0) = 0}$

$u_x(\pi,t) = X'(\pi) T(t) = 0 \Rightarrow \underline{X'(\pi) = 0}$

Hence we get the ODE

$$\begin{cases} X''(x) = \lambda\, X(x) \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

STEP 3: Boundary-value problem

Again, argue in terms of the sign of λ
(but this time slightly different!)

CASE 1: $\lambda > 0$

Then $\lambda = \omega^2$ for some $\omega > 0$

Then:

$X'' = \lambda\, X \Rightarrow X'' = \omega^2\, X \Rightarrow X'' - \omega^2\, X = 0$

Aux: $r^2 - \omega^2 = 0 \Rightarrow r^2 = \omega^2 \Rightarrow r = \pm\omega$

$\Rightarrow X(x) = A\, e^{\omega x} + B\, e^{-\omega x}$

$\Rightarrow X'(x) = A\, \omega\, e^{\omega x} - B\, \omega\, e^{-\omega x}$

$\Rightarrow X'(0) = A\,\omega\, e^{\omega 0} - B\,\omega\, e^{-\omega 0} = A\omega - B\omega = (A-B)\cancel{\omega} = 0$

$\Rightarrow B = A$

So $X(x) = A\, e^{\omega x} + A\, e^{-\omega x}$

$\Rightarrow X'(x) = A\omega\, e^{\omega x} - A\omega\, e^{-\omega x}$

$\Rightarrow X'(\pi) = A\omega\, e^{\omega\pi} - A\omega\, e^{-\omega\pi} = \cancel{A\omega}\,(e^{\omega\pi} - e^{-\omega\pi}) = 0$

$\Rightarrow e^{\omega\pi} - e^{-\omega\pi} = 0$

$\Rightarrow e^{\omega\pi} = e^{-\omega\pi}$

$\Rightarrow \omega\pi = -\omega\pi$

$\Rightarrow \omega = 0$

But then $\lambda = \omega^2 = 0 \Rightarrow\Leftarrow$ (since we assumed $\lambda > 0$)

CASE 2: $\lambda = 0$

Then $X'' = 0\,X \Rightarrow X''(x) = 0$

$\Rightarrow X(x) = Ax + B$

$\Rightarrow X'(x) = A$

$X'(0) = A = 0$, so $A = 0$ and $X(x) = 0x + B = B$

But notice that if $X(x) = B$, then automatically $X'(\pi) = 0$!

NEW: $\lambda = 0$ works and $X(x) = B$ is a solution!

CASE 3: $\lambda < 0$

Then $\lambda = -\omega^2$ for some $\omega > 0$

$X'' = \lambda X \Rightarrow X'' = -\omega^2 X \Rightarrow X'' + \omega^2 X = 0$

Aux: $r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm\, \omega i$

$X(x) = A \cos(\omega x) + B \sin(\omega x)$

$X'(x) = -A\omega \sin(\omega x) + B\omega \cos(\omega x)$

$$\begin{aligned} X'(0) &= -A\omega \sin(\omega 0) + B\omega \cos(\omega 0) \\ &= -A\omega\, 0 + B\omega\, 1 \\ &= B\cancel{\omega} = 0 \end{aligned}$$

(Cancel out ω since $\omega > 0$)

$\Rightarrow B = 0$

So $X(x) = A \cos(\omega x) + 0 \sin(\omega x) = A \cos(\omega x)$

$X'(x) = -A\omega \sin(\omega x)$

$$\begin{aligned} X'(\pi) &= -A\omega \sin(\pi\omega) = 0 \\ &\Rightarrow \sin(\pi\omega) = 0 \\ &\Rightarrow \pi\omega = \pi m \\ &\Rightarrow \omega = m \quad (m = 1, 2, \ldots) \end{aligned}$$

Answer: For every $m = 1, 2, \ldots$, we have a solution,

$X(x) = \cos(\omega x) = \cos(mx) \quad (m = 1, 2, \ldots)$

(Different from before!)

Conclusion: $\lambda = -m^2$ $(m = 1, 2, \dots)$

$X(x) = \cos(mx)$ $(m = 1, 2, \dots)$

Important remark: If you let $m = 0$ in the above, you get

$\lambda = -0^2 = 0$ and $X(x) = \cos(0x) = 1$, which is exactly Case 2 !

NEW: Actual conclusion: $\lambda = -m^2$ $(m = 0, 1, 2, \dots)$

$X(x) = \cos(mx)$ $(m = 0, 1, 2, \dots)$

Note: That's why later we'll sum from $m = 0$ to infinity instead from $m = 1$ to infinity

STEP 4: T equation

$$\frac{T'}{kT} = \lambda = -m^2$$

$\Rightarrow T' = -k\, m^2\, T$

$\Rightarrow T(t) = C\, e^{-m^2 kt}$

Note: This is also valid for $m = 0$, $T(t) = C$

Conclusion: For every $m = 0, 1, 2, \dots$

$u(x,t) = X(x)T(t) = C\, e^{-m^2 kt} \cos(mx)$

is a solution of our PDE

STEP 5: Linearity

Take linear combos

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} A_m e^{-m^2 kt} \cos(mx)}$$

STEP 6: Initial Condition

$$u(x,0) = \sum_{m=0}^{\infty} A_m \underbrace{e^{-m^2 k0}}_{1} \cos(mx)$$

$$x = \sum_{m=0}^{\infty} A_m \cos(mx)$$

This time we have a cosine problem!!!

$$\boxed{x = \sum_{m=0}^{\infty} A_m \cos(mx)}$$

WTF

This time: Can you write x as a linear combo of cosines?

YES, see Chapter 5

Note: Beware: the book (and others) here use $A_0/2$ instead of A_0, but in the end you should get the same expansion.

Next time: Actually figuring out how to calculate the coefficients! (Just based on neat linear algebra)

III- INHOMOGENEOUS PROBLEM

1) What if you had to solve?

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad (0 < x < \pi) \\ u(0,t) = 7, \ u(\pi,t) = 7 \\ u(x,0) = x^2 \\ u_t(x,0) = x \end{cases}$$

Trick: Let $\boxed{v(x,t) = u(x,t) - 7}$

Then $v_{tt} = c^2 v_{xx}$

$v(0,t) = 7-7 = 0$
$v(\pi,t) = 7 - 7 = 0$
$v(x,0) = x^2 - 7$
$v_t(x,0) = u_t(x,0) = x$

=> Solve

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0, \ v(\pi,t) = 0 \end{cases}$$

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = x^2 - 7 \\ v_t(x,0) = x \end{cases}$$

Then solve for v using the techniques from the previous lecture, and finally use

$u(x,t) = v(x,t) + 7$

2) **SAME** with $u_x(0,t) = 7$, $u_x(\pi,t) = 7$

$v(x,t) = u(x,t) - 7x$
$v(x,0) = x^2 - 7x$
$v_t(x,0) = x$

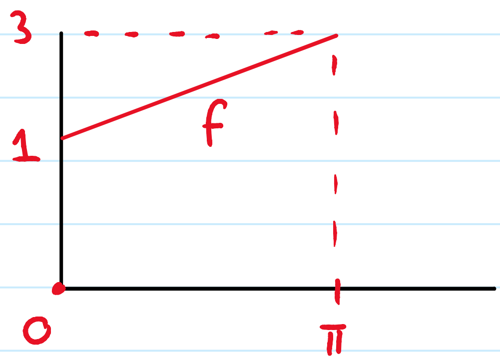
3) More interestingly:

Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 1,\ u(\pi,t) = 3 \\ u(x,0) = x^2 \\ u_t(x,0) = x \end{cases}$$

IDEA: Let $v(x,t) = u(x,t) - f(x)$

Where f is a linear function with $f(0) = 1$ and $f(\pi) = 3$



$$f(x) = \left(\frac{3-1}{\pi - 0}\right) x + 1 = \left(\frac{2}{\pi}\right) x + 1$$

$$v(x,t) = u(x,t) - \frac{2}{\pi} x - 1$$

$$v(0,t) = u(0,t) - 0 - 1 = 1 - 1 = 0$$

$$v(\pi,t) = u(\pi,t) - \frac{2}{\cancel{\pi}} \cancel{\pi} - 1 = 3 - 2 - 1 = 0$$

$$v(x,0) = u(x,0) - \frac{2}{\pi} x - 1 = x^2 - \frac{2}{\pi} x - 1$$

$$v_t(x,0) = u_t(x,0)$$

=> Solve

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0, \; v(\pi,t) = 0 \\ v(x,0) = x^2 - \frac{2}{\pi} x - 1 \\ v_t(x,0) = x \end{cases}$$

And use $u(x,t) = v(x,t) + \frac{2}{\pi} x + 1$

Note: Can *in theory* also solve $u_x(0,t) = 1$ with $u_x(\pi,t) = 3$ because you would subtract a function whose derivative is $\frac{2}{\pi} x + 1$, but your PDE will actually become inhomogeneous!

IV- WAVE EQUATION

What if you want to solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad (0 < x < \pi) \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0 \\ u(x,0) = x \\ u_t(x,0) = x^2 \end{cases}$$

STEPS 1 - 3: Same

STEP 4: Now we get the equation

$T''(t) = c^2 \lambda \; T(t)$

But $\lambda = -m^2$ with $m = 0, 1, 2, \ldots$ (from STEP 3)

If $m = 0$, then get $T''(t) = 0 \Rightarrow T(t) = A_0 + B_0 t$

$u(x,t) = X(x)\,T(t) = (A_0 + B_0 t)\cos(0x) = A_0 + B_0 t$

If $m = 1, 2, \ldots$, then get $T''(t) = c^2(-m^2)\,T(t) = -(mc)^2\,T(t)$

$\Rightarrow T(t) = A_m \cos(mct) + B_m \sin(mct)$

$u(x,t) = X(x)\,T(t) = [A_m \cos(mct) + B_m \sin(mct)] \cos(mx)$

$(m = 1, 2, \ldots)$

STEP 5: Linear combos:

$$u(x,t) = (A_0 + B_0\, t) + \sum_{m=1}^{\infty} [A_m \cos(mct) + B_m \sin(mct)] \cos(mx)$$

STEP 6: Initial condition

$$u(x,0) = (A_0 + B_0\, 0) + \sum_{m=1}^{\infty} [A_m \cos(0) + B_m \sin(0)] \cos(mx)$$

$$x = A_0 + \sum_{m=1}^{\infty} A_m \cos(mx)$$

$$x = \sum_{m=0}^{\infty} A_m \cos(mx)$$ SAME PROBLEM!

(This is because $A_0 = A_0 \cos(0x)$)

STEP 7: Initial velocity

$$u_t(x,t) = B_0 + \sum_{m=1}^{\infty} [-A_m (mc) \sin(mct) + B_m\, mc \cos(mct)] \cos(mx)$$

$$u_t(x,0) = B_0 + \sum_{m=1}^{\infty} [-A_m (mc) \sin(0) + B_m\, mc \cos(0)] \cos(mx)$$

$$x^2 = B_0 + \sum_{m=1}^{\infty} B_m mc \cos(mx)$$

$$= \tilde{B}_0 + \sum_{m=1}^{\infty} \tilde{B}_m \cos(mx)$$

$$= \sum_{m=0}^{\infty} \tilde{B}_m \cos(mx)$$ SAME PROBLEM!

Where $\tilde{B}_0 = B_0$, $\tilde{B}_m = B_m mc$

So first you'd find the coefficients $\tilde{B}_m$ and then you find B_m by using:

$B_0 = \tilde{B}_0$ and $B_m = \tilde{B}_m / (mc)$