LECTURE 21: FOURIER SERIES (I)

Thursday, November 14, 2019 1:47 PM

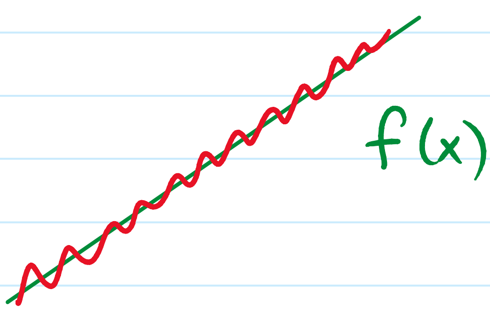
Today: We're FINALLY going to solve the problem that's been bugging us for a week now!

I- PRELUDE

GOAL: Given a function f(x) on $(0,\pi)$, find A_m such that:

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx) \qquad \text{(Fourier sine series)}$$

Intuitively: Want to write f in terms of sine waves



(Sort of like a Taylor series, but with sin instead of x^n)

Surprisingly, we can solve this using… Linear Algebra!

II- ORTHOGONALITY

Let u, v, w be vectors (in R^n)

Definition: u and v are **orthogonal** (= perpendicular) if $u \cdot v = 0$

Definition: $\{u,v,w\}$ is **orthogonal** if any two *different* vectors in that set are orthogonal

IMPORTANT FACT:

Suppose $\{u,v,w\}$ is orthogonal and $x = a\,u + b\,v + c\,w$, then:

$$a = \frac{x \cdot u}{u \cdot u} \qquad b = \frac{x \cdot v}{v \cdot v} \qquad c = \frac{x \cdot w}{w \cdot w}$$

(AMAZING, no row-reduction required; you have explicit formulas for the coefficients)

Why?

$$x \cdot u = (au + bv + cw) \cdot u = a\, u \cdot u + b\, \cancelto{0}{v \cdot u} + c\, \cancelto{0}{w \cdot u} = a\, u \cdot u$$

$$\Rightarrow a = \frac{x \cdot u}{u \cdot u}$$

(Similar for b and c)

Analogy: x hugs u (= $x \cdot u$) and then u is so happy that it hugs itself (= $u \cdot u$)

III- FOURIER SINE SERIES

What does orthogonality have to do with our problem?

Goal: $f(x) = \displaystyle\sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0,\pi)$

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It turns out that all the $\sin(mx)$ are orthogonal, but we first need to define the dot product of functions

Definition: $f \cdot g = \int_0^{\pi} f(x)\, g(x)\, dx$

Ex: $x \cdot x^2 = \int_0^{\pi} x\, x^2\, dx = \int_0^{\pi} x^3\, dx = \pi^4/4$

Note: Compare to usual dot product:

$(1,2,3) \cdot (4,5,6) = 1 \times 4 + 2 \times 5 + 3 \times 6 =$ Sum of products

Here $\int_0^{\pi} f(x)\, g(x)\, dx$ is an infinite sum of products

(Sum) (Product)

Fact: $\{\sin(x), \sin(2x), \ldots\} = \{\sin(mx) \mid m = 1, 2, \ldots\}$ is orthogonal

Why? Just need to check $\sin(mx) \cdot \sin(nx) = 0$ for $m \neq n$

That is, check $\int_0^{\pi} \sin(mx) \sin(nx)\, dx = 0$ (see HW 7)

CONSEQUENCE:

In particular, if $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$, by the above fact this implies:

$$A_m = \frac{f(x) \cdot \sin(mx)}{\sin(mx) \cdot \sin(mx)}$$ (f hugs $\sin(mx)$ and $\sin(mx)$ hugs itself)

$$= \frac{\int_0^{\pi} f(x) \sin(mx)\, dx}{}$$

$$\int_0^{\pi} \sin(mx)\sin(mx)\,dx$$

$$= \frac{\int_0^{\pi} f(x)\sin(mx)\,dx}{\int_0^{\pi} \sin^2(mx)\,dx}$$

$$= \frac{\int_0^{\pi} f(x)\sin(mx)\,dx}{\pi/2}$$

(see HW 7, $\pi/2$ because $(0,\pi)$ is half the interval $(-\pi,\pi)$)

=> IMPORTANT FACT:

If $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0,\pi)$, then

$$A_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$$

More generally, if $f(x) = \sum_{m=1}^{\infty} A_m \sin(\pi mx/l)$ on $(0,l)$ then

$$\boxed{A_m = \frac{2}{l}\int_0^{l} f(x)\sin(\pi mx/l)\,dx}$$

(Again, you hug f with $\sin(\pi m x/l)$ and you divide by l and don't forget about the 2)

IV - EXAMPLE

Example: Find A_m such that $x = \sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0,\pi)$

$$A_m = \frac{2}{\pi}\int_0^{\pi} x \sin(mx)\,dx \qquad \begin{array}{ll} u = x & dv = \sin(mx) \\ du = 1 & v = (-1/m)\cos(mx) \end{array}$$

$$= \frac{2}{\pi}\left(\left[\frac{-x\cos(mx)}{m}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(mx)}{m}\,dx\right)$$

$$= \frac{2}{\pi}\left(\frac{-\pi\cos(\pi m)}{m} + 0 + \left[\frac{1}{m^2}\sin(mx)\right]_0^{\pi}\right)$$

0

$$= \frac{2}{\cancel{\pi}}\;\frac{-\cancel{\pi}(-1)^m}{m}$$

$$= \frac{2}{m}(-1)^{m+1} \qquad \text{(Basically 2/m but alternates)}$$

$$x = 2\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x)$$

Consequence 1: If you let $x = \pi/2$ in the above formula, then

$\pi/2 = 2\sin(\pi/2) - \sin(\pi) + 2/3\sin(3\pi/2) - 1/2\sin(2\pi) + \dots$

$\pi/2 = 2 - 2/3 + 2/5 - 2/7 + \dots$

$\Rightarrow 1 - 1/3 + 1/5 - 1/7 \dots = \pi/4$ (WOW!)

Consequence 2:

Solve
$$\begin{cases} u_t = k\, u_{xx} & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x \end{cases}$$

So far, got: $u(x,t) = \sum_{m=1}^{\infty} A_m e^{-m^2 kt} \sin(mx)$

$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = x$

$A_m = 2/m \quad (-1)^{m+1}$

Solution: $u(x,t) = \sum_{m=1}^{\infty} 2/m\, (-1)^{m+1} e^{-m^2 kt} \sin(mx)$

V- DI METHOD

In practice, it's a pain to calculate the coefficients, except there's an NEAT method to deal with this :)

Example: $x^2 = \sum_{m=1}^{\infty} A_m \sin(mx) \quad$ on $(0,\pi)$

$$A_m = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(mx)\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(mx)\, dx$$

DI METHOD:

1. Make sign-table starting with + (and alternate signs)

2. Put x^2 on left-hand-side and differentiate the hell out of that until you get 0
3. Put $\sin(\pi m x)$ on the right-hand-side and anti-differentiate the hell out of that

	+	x^2	$\sin(mx)$
'	−	$2x$	$\dfrac{-\cos(mx)}{m}$
'	+	2	$\dfrac{-\sin(mx)}{m^2}$
'	−	0	$\dfrac{\cos(mx)}{m^3}$

$$A_m = \frac{2}{\pi}\left[+x^2\left(-\cos(mx)/m\right) - 2x\left(-\sin(mx)/m^2\right) + 2\cos(mx)/m^3\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\pi^2(-\cos(\pi m)/m) + 2\pi \cancelto{0}{\sin(\pi m)}/m^2 + 2\cos(\pi m)/m^3 - 0 + 0 - 2\cos(0)/m^3\right)$$

$$= \frac{2}{\pi}\left(-\pi^2(-1)^m/m + 2(-1)^m/m^3 - 2/m^3\right)$$

$$= 2\pi(-1)^{m+1}/m + 4/\pi(m^3)\left((-1)^m - 1\right)$$

$$= \begin{cases} -2\pi/m & \text{if } m \text{ is even} \\ 2\pi/m - 8/\pi(m^3) & \text{if } m \text{ is odd} \end{cases}$$