LECTURE 22: FOURIER SERIES (II)

Friday, November 15, 2019 6:19 PM

I- FOURIER COSINE SERIES

What if you want to write f as a cosine series?

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx) \text{ on } (0,\pi) \qquad \text{(beware: we start at } m = 0\text{)}$$

EVERYTHING we said about sin is also true about cos, and in particular:

$$A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(mx)\, dx \qquad (m = 1, 2, \dots)$$

WARNING:

$$A_0 = \frac{\int_0^{\pi} f(x) \cos(0x)\, dx}{\int_0^{\pi} \cos(0x) \cos(0x)\, dx} = \frac{\int_0^{\pi} f(x)\, dx}{\int_0^{\pi} 1\, dx}$$

$$A_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx \qquad (\textbf{NOT } 2/\pi)$$

Note: Book uses $A_0 / 2$, but defines A_0 differently.

$$\boxed{A_m = \frac{2}{\ell} \int_0^{\ell} f(x) \cos(\pi m x / l)\, dx \qquad A_0 = \frac{1}{\ell} \int_0^{\ell} f(x)\, dx}$$

Example: Write $x^3 = \sum_{m=0}^{\infty} A_m \cos(\pi m x)$ on $(0,1)$

Always isolate the case $m = 0$

$$A_0 = \frac{1}{1}\int_0^1 x^3\,dx = \boxed{\frac{1}{4}}$$

$$A_m = \frac{2}{1}\int_0^1 x^3\cos(\pi m x)\,dx$$

+	x^3	$\cos(\pi m x)$
-	$3x^2$	$\sin(\pi m x)/(\pi m)$
+	$6x$	$-\cos(\pi m x)/(\pi m)^2$
-	6	$-\sin(\pi m x)/(\pi m)^3$
+	0	$\cos(\pi m x)/(\pi m)^4$

$$\begin{aligned}A_m = 2\ \big[&1^3\,\underbrace{\sin(\pi m)}_{0}/(\pi m) + 3(1)^2\cos(\pi m)/(\pi m)^2 \\ &- 6(1)\underbrace{\sin(\pi m)}_{0}/(\pi m)^3 - 6\cos(\pi m)/(\pi m)^4 \\ &- 0 + 0 - 0 + 6\cos(0)/(\pi m)^4\big]\end{aligned}$$

$$= 6(-1)^m/(\pi m)^2 - 12(-1)^m/(\pi m)^4 + 12/(\pi m)^4$$

$$= 6(-1)^m/(\pi m)^2 + 12/(\pi m)^4\left((-1)^{m+1} + 1\right)$$

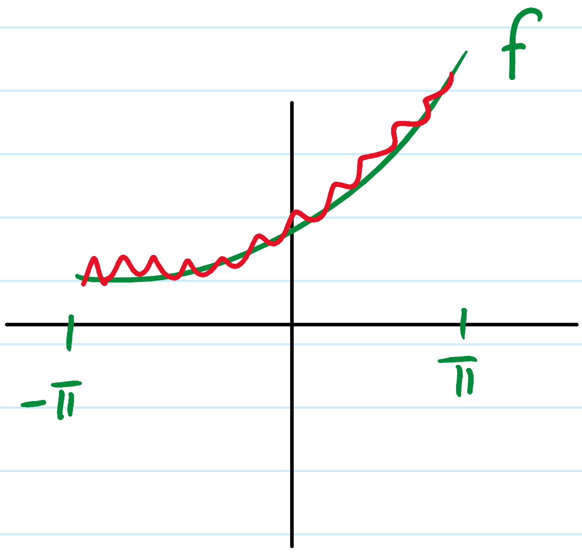
$$= \begin{cases} 6/(\pi m)^2 & \text{if } m \text{ is even} \\ -6/(\pi m)^2 + 24/(\pi m)^4 & \text{if } m \text{ is odd}\end{cases}$$

II- FULL FOURIER SERIES

What if you want to write f in terms of sin AND cos?

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx) \qquad \text{on } (-\pi, \pi)$$

(Notice the bigger interval!)



The beautiful thing is that we said so far is still true

Fact: $\{\sin(mx), \cos(mx) \mid m = 0, 1, 2, \ldots\}$ is orthogonal

(bigger set, but *still* orthogonal)

$$A_m = \frac{\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx}{\int_{-\pi}^{\pi} \cos(mx)\cos(mx)\,dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

(This time π, not $\pi/2$ because we're working on the FULL interval $(-\pi,\pi)$)

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

EXCEPTION: $m = 0$

$$A_0 = \frac{\int_{-\pi}^{\pi} f(x)\, 1\, dx}{\int_{-\pi}^{\pi} (1)(1)\, dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \quad \text{(still half of } A_m)$$

$B_0 = 0$ (by convention, since it corresponds to $\sin(0x) = 0$)

Example: Find the full Fourier series of $f(x) = x$ on $(-\pi,\pi)$

$$x = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)$$

$m = 0$

$$A_0 = \frac{\int_{-\pi}^{\pi} x\, dx}{\int_{-\pi}^{\pi} 1\, dx} = \frac{0}{2\pi} = 0$$

$B_0 = 0$

$m = 1, 2, \dots$

$$A_m = \frac{\int_{-\pi}^{\pi} x\cos(mx)\, dx}{\int_{-\pi}^{\pi} \cos(mx)\cos(mx)\, dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(mx)}_{\text{ODD}}\, dx = 0$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\sin(mx)}_{\text{EVEN}}\, dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(mx)\, dx$$

$B_m = \frac{2}{m}(-1)^{m+1}$ (from last time)

So $x = \sum_{m=1}^{\infty} 0 \cos(mx) + \frac{2}{m}(-1)^{m+1} \sin(mx)$ $(A_0 = B_0 = 0)$

$$x = \sum_{m=1}^{\infty} \frac{2}{m}(-1)^{m+1} \sin(mx)$$

Fact: If f(x) is odd on $(-\pi,\pi)$, then the full FS is a sine series

" even " cosine series

(In practice have to calculate BOTH A_m and B_m)

III- COMPLEX FOURIER SERIES

To make things more complex, we can also consider complex Fourier series

Quick recap:

1. Complex numbers: $2 + 3i$
2. $e^{ix} = \cos(x) + i \sin(x)$
3. $\overline{a+bi} = a - bi$
4. In particular, $\overline{e^{ix}} = e^{-ix}$
5. $(a-bi)(a+bi) = a^2 - (bi)^2 = a^2 + b^2$
6. $Re(a+b) = a$, $Im(a+bi) = b$

Also, for *complex* functions f and g on $(-\pi,\pi)$

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

(Otherwise $f \cdot f$ isn't positive any more)

Goal: Write any function $f(x)$ on $(-\pi,\pi)$ as:

$$f(x) = \sum_{M=-\infty}^{\infty} C_m \, e^{imx}$$

(**Note:** Here the sum goes from $-\infty$ to ∞ and that's because $e^{imx} \neq e^{i(-m)x}$, whereas before $\cos(mx) = \cos(-mx)$)

Fact: $\{e^{imx} \mid m = \dots, -1, 0, 1, \dots\}$ is orthogonal

$$C_m = \frac{f \cdot e^{imx}}{e^{imx} \cdot e^{imx}} = \frac{\int_{-\pi}^{\pi} f(x) \overline{e^{imx}} \, dx}{\int_{-\pi}^{\pi} e^{imx} \overline{e^{imx}} \, dx}$$

$$= \frac{\int_{-\pi}^{\pi} f(x) e^{-imx} \, dx}{\int_{-\pi}^{\pi} e^{imx} e^{-imx} \, dx} = \frac{\int_{-\pi}^{\pi} f(x) e^{-imx} \, dx}{\int_{-\pi}^{\pi} 1 \, dx}$$

$$\boxed{C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx} \quad \text{(ALSO valid if } m = 0\text{)}$$

In general: $C_m = \dfrac{1}{2\ell} \displaystyle\int_{-\ell}^{\ell} f(x) e^{-i\pi m x / l} \, dx$

Example: Complex Fourier series of $f(x) = e^x$ on $(-\pi,\pi)$

$$C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-imx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-im)x} \, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(1-im)x}}{(1-im)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left(\frac{e^{(1-im)\pi} - e^{(1-im)(-\pi)}}{(1-im)} \right)$$

$$= \frac{1}{2\pi} \frac{1}{(1-im)} \left(e^{\pi} e^{-\pi m i} - e^{-\pi} e^{\pi m i} \right)$$

Note: $e^{\pi m i} = \cos(\pi m) + i \cancelto{0}{\sin(\pi m)} = (-1)^m$

$$e^{-\pi m i} = \cos(-\pi m) + i \underbrace{\cancel{\sin(\pi m)}}_{0} = \cos(\pi m) = (-1)^m$$

$$= \frac{1}{\pi} \frac{1}{(1-im)} (-1)^m \frac{e^{\pi} - e^{-\pi}}{2}$$

$$= \frac{1}{\pi} \frac{1}{1-im} (-1)^m \sinh(\pi)$$

But $\dfrac{1}{1-im} = \dfrac{1}{1-im} \dfrac{1+im}{1+im} = \dfrac{1+im}{1^2+m^2} = \dfrac{1+im}{m^2+1}$

$$\boxed{C_m = \frac{(-1)^m \sinh(\pi)}{\pi(m^2+1)} (1+im)}$$

$$e^x = \sum_{M=-\infty}^{\infty} \frac{(-1)^m \sinh(\pi)(1+im)}{\pi(m^2+1)} e^{imx}$$

Will see a *really* cool application of that next time!

(Kinda) Cool Application:

Note: $C_m = \dfrac{(-1)^m \sinh(\pi)}{\pi(m^2+1)} \quad (1 + im)$

On the one hand, $\mathrm{Re}(C_m) = \dfrac{(-1)^m \sinh(\pi)}{\pi(m^2+1)}$

On the other hand,

$$\mathrm{Re}(C_m) = \mathrm{Re}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\, \overset{\cos(-mx) + i\sin(-mx)}{e^{-imx}}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cos(-mx)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cos(mx)\, dx = \frac{A_m}{2}$$

$\Rightarrow A_m = \dfrac{2(-1)^m \sinh(\pi)}{\pi(m^2+1)}$

$\left(A_0 = \dfrac{\sinh(\pi)}{\pi}\right)$

Similarly, by taking imaginary parts we get

$B_m = \dfrac{-2m(-1)^m \sinh(\pi)}{\pi(m^2+1)} \qquad (B_0 = 0 \text{ by convention})$

So with those values of A_m and B_m, we get

$$e^x = \sum_{M=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)$$

(So get full Fourier series from complex Fourier series, **WOW**)