LECTURE 23: FOURIER SERIES (III)

Tuesday, November 19, 2019 1:35 PM

Welcome to our final installment on Fourier series! Today we'll discuss some general properties of Fourier series

I- CONVERGENCE OF FOURIER SERIES

$$f(x) = \underbrace{\sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)}_{\mathcal{F}(x)} \quad \text{on } (-\pi,\pi)$$

Question: When is $=$ an actual equality? That is, when is f equal to its Fourier series $\mathcal{F}$?

Why bother?

Crazy Fact: There is a function f for which $\mathcal{F}(x) = \pm\infty$ EVERYWHERE!!!

Luckily, we have the following fact:

FACT: 1) If f is continuous at x, then $\mathcal{F}(x) = f(x)$

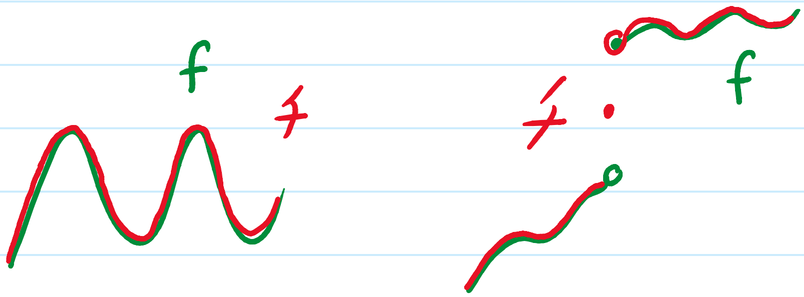
2) If f has a jump at x, then

$$\mathcal{F}(x) = 1/2\,(f(x^-) + f(x^+)) \qquad \text{(Average of jumps)}$$

(Where $f(x^-)$ and $f(x^+)$ are the left and right limits of f at x)

(Note: Here by convergence we mean pointwise convergence)

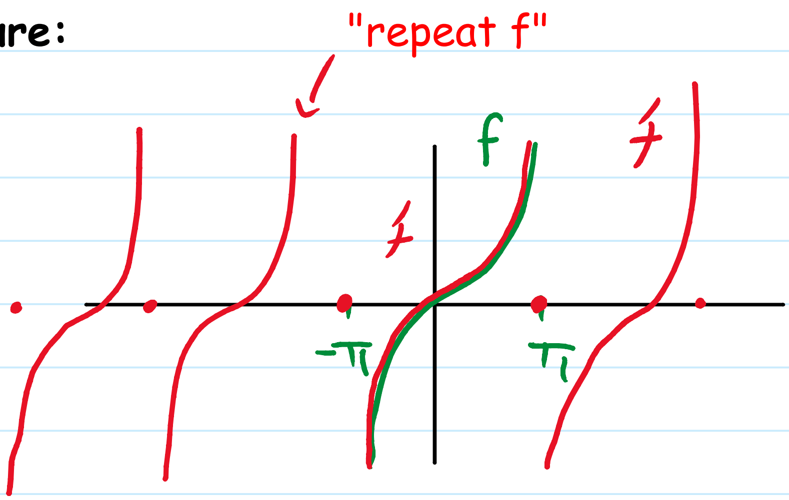
Picture



Example: Let $f(x) = x^3$ on $(-\pi,\pi)$, draw the graph of $\mathcal{F}(x)$ on all of R

Notice: $\mathcal{F}(x)$ is periodic, so need to first "periodify" f and then apply the rules above

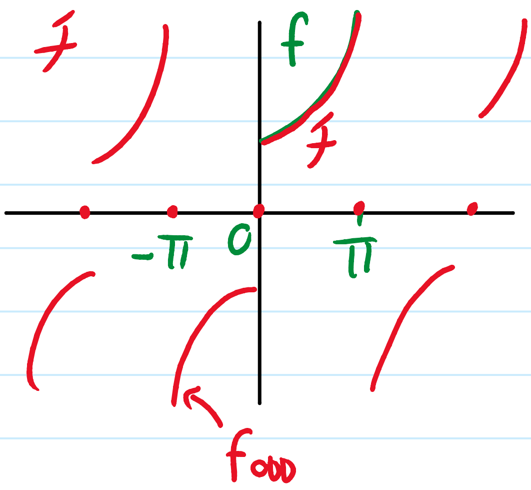
Picture:



Example: $f(x) = x^2 + 1$ on $(0,\pi)$, draw the graph of the Fourier sine series of f on R.

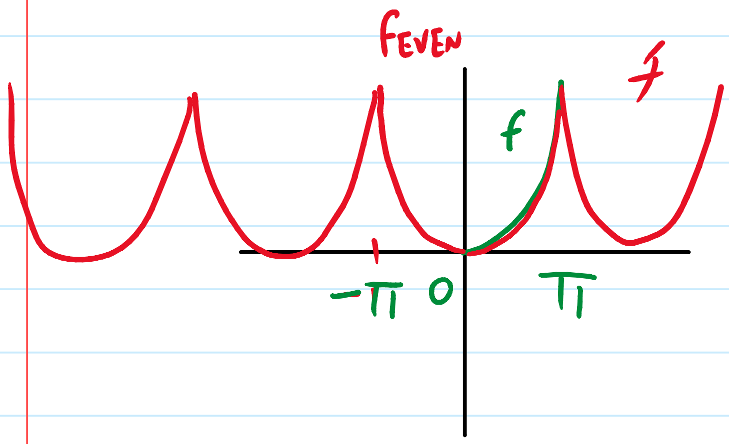
Notice the Fourier sine series is odd (because linear combo of sines), so first **ODDIFY** f and apply the previous rule

Picture:



Example: $f(x) = x^3$ but with cosine series

This time evenify



Note: The question of convergence of Fourier series is extremely delicate; there's even a whole math subject called Harmonic Analysis just dedicated to the question of convergence of Fourier series. Luckily for the PDEs in this course, all the functions are continuous, so we always have equality.

II- PARSEVAL'S EQUALITY

Once upon a time, there was this mighty knight called Parseval. He was in the quest of the holy grail, but who knew that he would find it in PDEs?

A) MOTIVATION

Recall:

1) $||u||^2 = u \cdot u$

2) Pythagorean Theorem:

If $u \perp v$, then $||u+v||^2 = ||u||^2 + ||v||^2$

If $\{u,v,w\}$ is orthogonal, then $||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2$

Well, isn't a Fourier series isn't just an infinite sum of orthogonal vectors?

B) THE FORMULA

Suppose $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0,\pi)$

Consequence:
$$||f||^2 = \left|\left| \sum_{m=1}^{\infty} A_m \sin(mx) \right|\right|^2$$
$$= \sum_{m=1}^{\infty} ||A_m \sin(mx)||^2$$
$$= \sum_{m=1}^{\infty} |A_m|^2 \, ||\sin(mx)||^2$$

But $||\sin(mx)||^2 = \sin(mx) \cdot \sin(mx) = \int_0^{\pi} \sin^2(mx)\, dx = \pi/2$

And $||f||^2 = \int_0^{\pi} (f(x))^2\, dx$

So ultimately we get:

$$\int_0^{\pi} (f(x))^2\, dx = \pi/2 \sum_{m=1}^{\infty} |A_m|^2$$

Fact: PARSEVAL'S IDENTITY

If $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$, then

$$\sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2\, dx$$

(Derive, don't memorize)

C) EXAMPLE 1

Warning: This is literally the most exciting fact you'll see in your life!

Example: $f(x) = x$ on $(0,\pi)$

Found $A_m = \frac{2}{m}(-1)^m$

$$\sum_{m=1}^{\infty} |A_m|^2 = \sum_{m=1}^{\infty} \left| \frac{2}{m} \overbrace{(-1)^m}^{1} \right|^2 = \sum_{m=1}^{\infty} \frac{4}{m^2} = 4 \sum_{m=1}^{\infty} \frac{1}{m^2}$$

On the other hand,

$$\int_0^{\pi} (f(x))^2 \, dx = \int_0^{\pi} x^2 \, dx = \frac{\pi^3}{3}$$

So Parseval says:

$$4 \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{2}{\pi} \frac{\pi^3}{3} = \frac{2}{3} \pi^2$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{4} \frac{2}{3} \pi^2 = \frac{\pi^2}{6}$$

WOOOOOW!!!

D) EXAMPLE 2

Note: The same thing is true for cosine series, but because

$||\cos(0x)||^2 = \cos(0x).\cos(0x) = \int_0^{\pi} 1 \, dx = \pi$, we have

$$\boxed{\text{If } f(x) = \sum_{m=0}^{\infty} A_m \cos(mx) \text{ on } (0,\pi) \text{, then } \quad 2|A_0|^2 + \sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 \, dx}$$

Example: $f(x) = x^2$ on $(0,\pi)$

$A_0 = \frac{\pi^2}{3}$ and $A_m = \frac{4(-1)^m}{m^2}$

$$2|A_0|^2 + \sum_{M=1}^{\infty} |A_m|^2 = \frac{2}{\pi}\int_0^{\pi} (x^2)^2\, dx$$

$$2\,\frac{\pi^4}{9} + \sum_{M=1}^{\infty} \frac{16}{m^4} = \frac{2}{\pi}\,\frac{\pi^5}{5} = \frac{2\pi^4}{5}$$

$$\sum_{M=1}^{\infty} \frac{16}{m^4} = \frac{2}{5}\pi^4 - \frac{2}{9}\pi^4 = \frac{8}{45}\pi^4$$

$$\sum_{M=1}^{\infty} \frac{1}{m^4} = \frac{1}{16}\,\frac{8}{45}\,\pi^4$$

$$\sum_{M=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90}$$

WOW!

Note: The sum of $1/m^3$ is still an open question

E) EXAMPLE 3

Well, the same thing also works for complex Fourier series!

Note: Here $f \cdot f = \int_{-\pi}^{\pi} f(x)\,\overline{f(x)}\, dx = \int_{-\pi}^{\pi} |f(x)|^2\, dx$

$$||e^{imx}||^2 = \int_{-\pi}^{\pi} e^{imx} e^{-imx} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$\boxed{\text{Fact: If } f(x) = \sum_{m=-\infty}^{\infty} C_m e^{imx} \text{ on } (-\pi,\pi) \text{, then } \sum_{m=-\infty}^{\infty} |C_m|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx}$$

Example: $f(x) = e^x$ on $(-\pi,\pi)$

Found:

$$C_m = \frac{1}{\pi} \frac{1}{1-im} (-1)^m \sinh(\pi)$$

$$|C_m|^2 = \frac{1}{\pi^2} \frac{1}{|1-im|^2} |(-1)^m|^2 \, |\sinh(\pi)|^2 = \frac{\sinh^2(\pi)}{\pi^2 (m^2+1)}$$

$(|a+bi|^2 = a^2 + b^2)$

$$\sum_{m=-\infty}^{\infty} |C_m|^2 = \sum_{m=-\infty}^{\infty} \frac{\sinh^2(\pi)}{\pi^2 (m^2+1)} = \frac{\sinh^2(\pi)}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{m^2+1}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|e^x|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{2x}\,dx$$

$$= \frac{1}{2\pi}\,\frac{e^{2\pi}-e^{-2\pi}}{2} = \frac{1}{2\pi}\sinh(2\pi)$$

So Parseval says:

$$\frac{\sinh^2(\pi)}{\pi^{\cancel{2}}}\sum_{M=-\infty}^{\infty}\frac{1}{m^2+1} = \frac{1}{2\cancel{\pi}}\sinh(2\pi)$$

$$\boxed{\sum_{M=-\infty}^{\infty}\frac{1}{m^2+1} = \frac{\pi}{2}\,\frac{\sinh(2\pi)}{\sinh^2(\pi)}}$$ WOOOW!

Note: Using symmetry, can clean this up to get

$$\sum_{M=1}^{\infty}\frac{1}{m^2+1} = \frac{\pi}{4}\left(\frac{\sinh(2\pi)}{\sinh^2(\pi)}\right)-\frac{1}{2}$$