LECTURE 24: LAPLACE'S EQUATION

Monday, November 4, 2019   12:47 PM

Welcome to the final equation of our course: Laplace's Equation! This is a PDE with lots of nice properties and miracles, and will take you all the way up to 112B.

Today: As our 4^{th} example of separation of variables, let's separate variables with Laplace's equation

I- SETTING

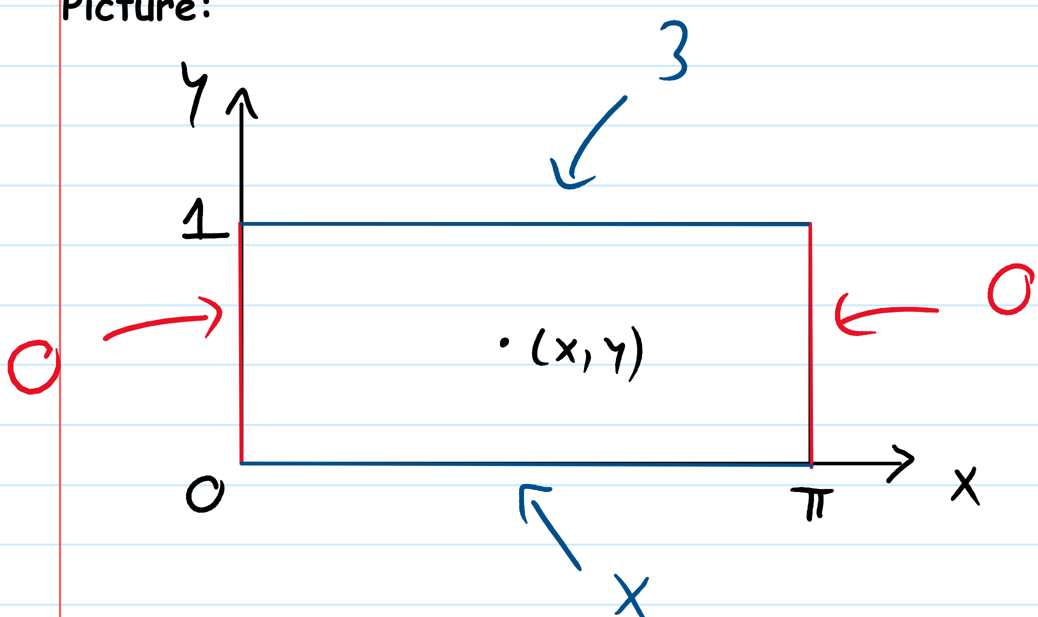
Here $u = u(x,y)$

(no more time dependence, but instead have boundary conditions)

Example: This time solve

$$\begin{cases} u_{xx} + u_{yy} = 0 & (0 < x < \pi,\ 0 < y < 1) \\ u(0,y) = 0 & u(\pi,y) = 0 \\ u(x,0) = x & u(x,1) = 3 \end{cases}$$

Picture:



Interpretation: $u(x,y)$ is the temperature of a 2D metal plate at (x,y), but after a looooong time (think heat equation but with $u_t = 0$)

II- SEPARATION OF VARIABLES

STEP 1: Separation of variables

1) Suppose:

$$\boxed{u(x,y) = X(x)\,Y(y)} \qquad (*)$$

2) Plug $(*)$ into $u_{xx} + u_{yy} = 0$

$$(X(x)\,Y(y))_{xx} + (X(x)Y(y))_{yy} = 0$$

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$$X''(x)\,Y(y) = -X(x)\,Y''(y)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

$\Rightarrow$ $X''(x) = \lambda X(x)$

And $Y''(y) = -\lambda Y(y)$

STEP 2: X(x) equation

Warning: You don't *always* choose the X equation here, you have to pick one the one with 0 boundary conditions!

Here $u(0,y) = 0$ and $u(\pi,y) = 0$ suggests you choose X (see below). Sometimes have to pick Y

So far: $X''(x) = \lambda X(x)$

Now use the boundary conditions:

$u(0,y) = 0 \Rightarrow X(0)\cancel{Y(y)} = 0 \Rightarrow \underline{X(0) = 0}$

Similarly $u(\pi,y) = 0 \Rightarrow X(\pi)\cancel{Y(y)} = 0 \Rightarrow \underline{X(\pi) = 0}$

Hence we get the ODE

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

STEP 3: 3 CASES

SKIP

Conclusion: $\lambda = -m^2$ $(m = 1, 2, \ldots)$

$X(x) = \sin(mx)$ $(m = 1, 2, \ldots)$

WARNING: Sometimes λ might be $-\lambda$
(depending on whether you're dealing with X or Y)
In any case, at the end, you should get sin or cos

STEP 4: Y equation

$Y'' = -\lambda Y$

$\Rightarrow Y'' = -(-m)^2 Y$

$\Rightarrow Y'' = m^2 Y \qquad (r^2 = m^2 \Rightarrow r = \pm m)$

$\Rightarrow Y(y) = A e^{my} + B e^{-my}$

NEW!!!!

For the wave equation we got $A \cos(mct) + B \sin(mct)$ and what was nice is that for $t = 0$ we had $A(1) + B(0) = A$.
It would be great if we had functions with a similar feature:

Recall: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Cosh and sinh are just linear combinations of exponential functions. So any linear combo of e^x and e^{-x} is also a linear combo of cosh(x) and sinh(x):

$$\begin{aligned} A e^x + B e^{-x} &= A(\cosh(x) + \sinh(x)) + B(\cosh(x) - \sinh(x)) \\ &= \underbrace{(A+B)}_{\tilde{A}} \cosh(x) + \underbrace{(A-B)}_{\tilde{B}} \sinh(x) \\ &= \tilde{A} \cosh(x) + \tilde{B} \sinh(x) \end{aligned}$$

Upshot: $Y(y) = A e^{my} + B e^{-my} = \tilde{A} \cosh(my) + \tilde{B} \sinh(my)$

for **different** (but still arbitrary) $\tilde{A}$ and $\tilde{B}$

Conclusion: For every $m = 1, 2, \ldots$

$u(x,y) = X(x)Y(y) = (A \cosh(my) + B \sinh(my)) \sin(mx)$
is a solution of our PDE

STEP 5: Linearity

$$\boxed{u(x,y) = \sum_{m=1}^{\infty} [A_m \cosh(my) + B_m \sinh(my)] \sin(mx)}$$

STEP 6: Remaining boundary conditions

Use $u(x,0) = x$ and $u(x,1) = 3$

$$u(x,0) = \sum_{m=1}^{\infty} [A_m \underbrace{\cosh(m0)}_{1} + B_m \cancel{\sinh(m0)}^{\,0}] \sin(mx)$$

$$x = \sum_{m=1}^{\infty} A_m \sin(mx) \quad \text{on } (0,\pi)$$

(Again, notice how elegant it is! If we didn't choose cosh and sinh, we could have a mess with exponential functions)

(Before we were stuck, but now we're Fourierxperienced)

$$A_m = \frac{2}{\pi} \int_0^{\pi} x \sin(mx)\, dx = \ldots = \frac{2}{m} (-1)^m$$

STEP 7: $u(x,1) = 3$

$$u(x,1) = \sum_{m=1}^{\infty} [A_m \cosh(m1) + B_m \sinh(m1)] \sin(mx)$$

$$= \sum_{M=1}^{\infty} \underbrace{[A_m \cosh(m) + B_m \sinh(m)]}_{\tilde{B}_M} \sin(mx)$$

$$= \sum_{M=1}^{\infty} \tilde{B}_m \sin(mx) \quad = \quad 3$$

Where $\tilde{B}_m = A_m \cosh(m) + B_m \sinh(m)$

$$\tilde{B}_m = \frac{2}{\pi} \int_0^{\pi} 3 \sin(mx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{-3 \cos(mx)}{m} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \frac{3}{m} [-\cos(\pi m) + 1]$$

$$= \frac{6}{\pi m} [(-1)^{m+1} + 1]$$

(= 0 if m is even, $12/(\pi m)$ if m is odd)

So $A_m \cosh(m) + B_m \sinh(m) = 6/\pi m \quad [(-1)^{m+1} + 1]$

$$\Rightarrow B_m = \frac{(6/\pi m)\ [(-1)^{m+1}+1] - A_m \cosh(m)}{\sinh(m)}$$

$$= \frac{(6/\pi m)\ [(-1)^{m+1}+1] - (2/m)\ (-1)^m \cosh(m)}{\sinh(m)}$$

STEP 8: CONCLUSION

$$\boxed{u(x,y) = \sum_{M=1}^{\infty} [A_m \cosh(my) + B_m \sinh(my)] \sin(mx)}$$

$$A_m = \frac{2}{m} (-1)^m$$

$$B_m = \frac{(6/\pi m)\ [(-1)^{m+1}+1] + (2/m)\ (-1)^{m+1} \cosh(m)}{\sinh(m)}$$

$$= \begin{cases} -(2/m)\coth(m) & \text{if m even} \\ 12/(\pi m)\,\mathrm{csch}(m) + (2/m)\coth(m) & \text{if m odd} \end{cases}$$

Interpretation: Temperature oscillates in the x direction (because of sin(mx) term) but is grows/decays exponentially in the y direction (because of cosh and sinh terms)

III- VARIATIONS

Example:

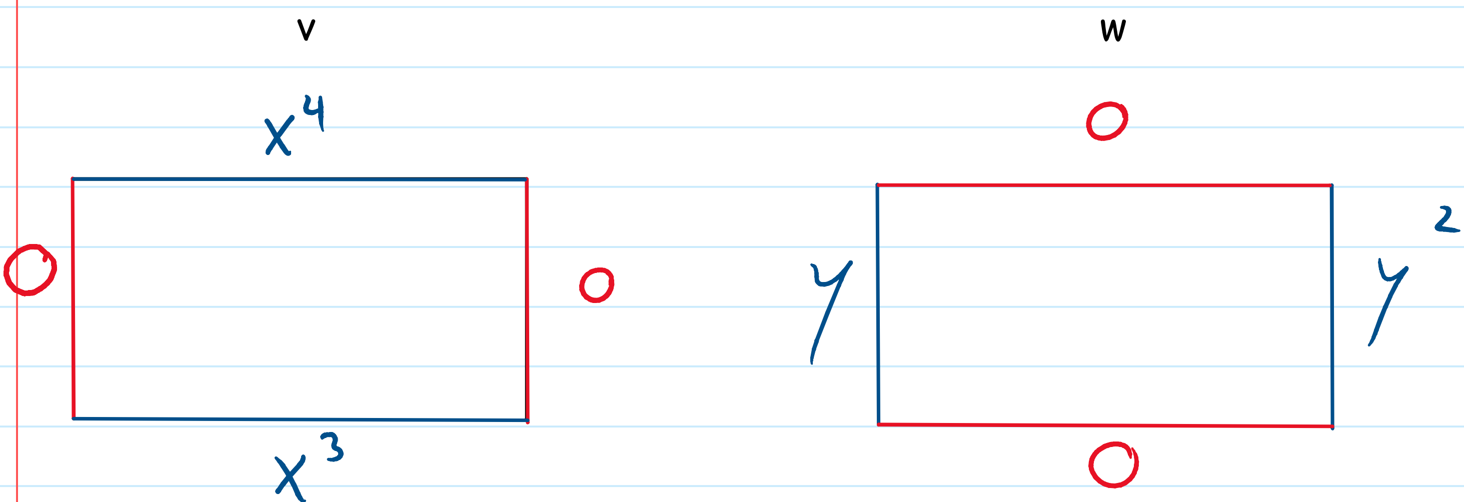
$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = y \qquad u(\pi,y) = y^2 \\ u(x,0) = x^3 \qquad u(x,1) = x^4 \end{cases}$$

Oh no! None of the terms are 0 !

Trick: Solve

$$\begin{cases} v_{xx} + v_{yy} = 0 \\ v(0,y) = 0 \quad v(\pi,y) = 0 \\ v(x,0) = x^3 \quad v(x,1) = x^4 \end{cases} \quad \text{and} \quad \begin{cases} w_{xx} + w_{yy} = 0 \\ w(0,y) = y \quad w(\pi,y) = y^2 \\ w(x,0) = 0 \quad w(x,1) = 0 \end{cases}$$

Picture:



Solve those separately, and then use $u = v + w$

(**Note**: For v equation, you do X first just like above, but for w equation you do Y first!)

Example: (Optional) Can in theory also solve

$$u_{xx} + u_{yy} + u_{zz} = 0$$

But beyond the scope of this course (might be done in 112B)

Get for instance (depending on boundary conditions)

$$u(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \sinh\left(\sqrt{m^2+n^2}\; x\right) \sin(my)\sin(nz)$$

$$A_{mn} = \frac{\int_0^{\pi}\int_0^{\pi} f(y,z)\,(\sin(my)\sin(nz))\,dy\,dz}{\int_0^{\pi}\int_0^{\pi} (\sin(my)\sin(nz))^2\,dy\,dz} \qquad \text{(Hugging)}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\pi}\int_0^{\pi} f(y,z)\sin(my)\sin(nz)\,dy\,dz$$

(**DOUBLE** Fourier series, **WHOA**)