LECTURE 25 - FUNDAMENTAL SOLUTION OF LAPLACE

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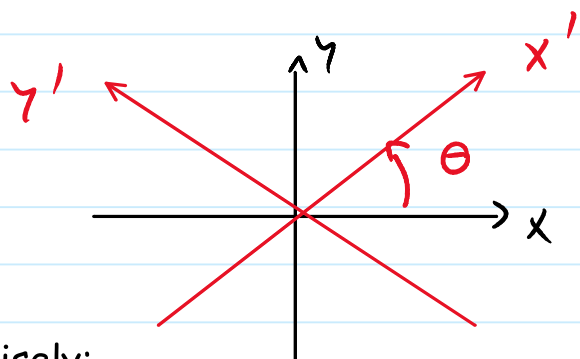
Today: Derive the fundamental solution of Laplace's equation (just like we did for the heat equation).

I- ROTATION INVARIANCE

Suppose $u = u(x,y)$ solves $u_{xx} + u_{yy} = 0$ on R^2

Important Fact: u is invariant under rotations

That is, if you rotate the plane by θ radians, then u (in the new variables) still solves Laplace's equation



More precisely:

Let θ be fixed (think $\theta = \pi/4$) and define:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ (Rotation matrix)

$$\begin{cases} x' = \cos(\theta)\, x - \sin(\theta)\, y \\ y' = \sin(\theta)\, x + \cos(\theta)\, y \end{cases}$$

Then $\boxed{u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0}$

Why? Use the Chen Lu!

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= u_{x'} \cos(\theta) + u_{y'} \sin(\theta) \qquad (*)$$

$$u_{xx} = \frac{\partial u_x}{\partial x}$$

$$= \frac{\partial u_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u_x}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= (u_{x'} \cos(\theta) + u_{y'} \sin(\theta))_{x'} \cos(\theta) + (u_{x'} \cos(\theta) + u_{y'} \sin(\theta))_{y'} \sin(\theta)$$

(θ is FIXED)

$$= u_{x'x'} \cos^2(\theta) + 2\, u_{x'y'} \cos(\theta)\sin(\theta) + u_{y'y'} \sin^2(\theta)$$

Similarly:

$$u_{yy} = u_{x'x'} \sin^2(\theta) - 2u_{x'y'} \cos(\theta)\sin(\theta) + u_{y'y'} \cos^2(\theta)$$

Therefore:

$$u_{xx} + u_{yy} = u_{x'x'} (\cos^2(\theta) + \sin^2(\theta)) + \cancel{2\, u_{x'y'} \cos(\theta)\sin(\theta)}$$

$$\cancel{-2u_{x'y'} \cos(\theta)\sin(\theta)} + u_{y'y'} (\sin^2(\theta) + \cos^2(\theta))$$

$$= u_{x'x'} + u_{y'y'}$$

Hence $u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$

Note:

1) This is also true in higher dimensions, if you replace rotations by orthogonal matrices ($O^T O = I$)

2) Therefore, it is natural to look for radial solutions, that is, solutions of the form $u(x,y) = v(r)$ with $r = \sqrt{x^2 + y^2}$ (see HW 8), but we'll pursue a different way

II- POLAR COORDINATES

Instead, we will use a coordinate system that is natural for rotations... polar coordinates!

Problem: Suppose you define

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$$

Then what does $u_{xx} + u_{yy} = 0$ become in terms of r and θ ?

Again, use the Chen Lu!

Note: $r = \sqrt{x^2 + y^2}$, so

$$\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{\cancel{r}\cos(\theta)}{\cancel{r}} = \cos(\theta)$$

$$\frac{\partial r}{\partial x} = \cos(\theta)$$

Similarly:

$$\frac{\partial r}{\partial y} = \sin(\theta)$$

And using $\theta = \tan^{-1}(y/x)$, have

$$\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r}$$

Now we are ready to Chen Lu!

$$u_x = \frac{\partial u}{\partial x}$$

$$= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= u_r \cos(\theta) + u_\theta \left(-\frac{\sin(\theta)}{r}\right)$$

$$u_{xx} = (u_x)_x$$

$$= \frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \underbrace{\left(u_r \cos(\theta) - u_\theta \frac{\sin(\theta)}{r}\right)_r \cos(\theta)}_{A} + \underbrace{\left(u_r \cos(\theta) - u_\theta \frac{\sin(\theta)}{r}\right)_\theta \left(\frac{-\sin(\theta)}{r}\right)}_{B}$$

$$= A + B$$

WARNING: Here r and θ are NOT fixed, so have to use the

Product rule!

$$A = \left(u_{rr}\cos(\theta) - u_{\theta r}\frac{\sin(\theta)}{r} + u_\theta \frac{\sin(\theta)}{r^2} \right) \cos(\theta)$$

$$= u_{rr}\cos^2(\theta) - u_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r} + u_\theta \frac{\sin(\theta)\cos(\theta)}{r^2}$$

$$B = \left(u_{r\theta}\cos(\theta) - u_r \sin(\theta) - u_{\theta\theta}\frac{\sin(\theta)}{r} - u_\theta \frac{\cos(\theta)}{r} \right)\left(-\frac{\sin(\theta)}{r} \right)$$

$$= -u_{r\theta}\frac{\cos(\theta)\sin(\theta)}{r} + u_r \frac{\sin^2(\theta)}{r} + u_{\theta\theta}\frac{\sin^2(\theta)}{r^2} + u_\theta \frac{\sin(\theta)\cos(\theta)}{r^2}$$

$A + B$

$$= u_{rr}\cos^2(\theta) - u_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r} + u_\theta \frac{\sin(\theta)\cos(\theta)}{r^2}$$

$$- u_{r\theta}\frac{\cos(\theta)\sin(\theta)}{r} + u_r \frac{\sin^2(\theta)}{r} + u_{\theta\theta}\frac{\sin^2(\theta)}{r^2} + u_\theta \frac{\sin(\theta)\cos(\theta)}{r^2}$$

$$= u_{rr}\cos^2(\theta) - 2u_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r} + 2u_\theta \frac{\sin(\theta)\cos(\theta)}{r^2} + u_r \frac{\sin^2(\theta)}{r} + u_{\theta\theta}\frac{\sin^2(\theta)}{r^2}$$

Similar formula for u_{yy} (see below)

Combine:

$u_{xx} + u_{yy} =$

$$u_{rr}\cos^2(\theta) - \cancel{2u_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r}} + \cancel{2u_\theta\frac{\sin(\theta)\cos(\theta)}{r^2}} + u_r\frac{\sin^2(\theta)}{r} + u_{\theta\theta}\frac{\sin^2(\theta)}{r^2} \quad +$$

$$u_{rr}\sin^2(\theta) + \cancel{2u_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r}} - \cancel{2u_\theta\frac{\sin(\theta)\cos(\theta)}{r^2}} + u_r\frac{\sin^2(\theta)}{r} + u_{\theta\theta}\frac{\cos^2(\theta)}{r^2}$$

$$= u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

FACT: [POLAR LAPLACE] (DO NOT MEMORIZE)

$$\boxed{u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}}$$

In particular, Laplace's equation in polar coordinates becomes

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0 \quad (*)$$

III- FUNDAMENTAL SOLUTION

How to obtain a solution from this?

If u is radial, then $u_\theta = 0$, so $u_{\theta\theta} = 0$, and $(*)$ becomes:

$$u_{rr} + \frac{u_r}{r} = 0 \text{ (an ODE!)}$$

$$u_{rr} = \frac{-u_r}{r}$$

$$\frac{u_{rr}}{u_r} = -\frac{1}{r}$$

$(\ln|u_r|)' = -1/r$

$\Rightarrow \ln|u_r| = -\ln(r) + C$

$\Rightarrow |u_r| = e^{-\ln(r)+C} = \dfrac{e^C}{e^{\ln(r)}}$

$\Rightarrow u_r = \dfrac{+/-\ e^C}{r} = \dfrac{C}{r}$

$\Rightarrow u = C\ln(r) + C'$

$\boxed{u(x,y) = C\ln\left(\sqrt{x^2+y^2}\right) + C' \text{ solves } u_{xx} + u_{yy} = 0}$ (do NOT memorize)

Aside: It turns out that, among all of them, there is one of them is most important namely the one with $C = -1/(2\pi)$ and $C' = 0$

$S(x,y) = -\dfrac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)$ is the fundamental solution of $u_{xx} + u_{yy} = 0$

Why fundamental? Because can build up other solutions from this!

Fun Fact: A solution of $u_{xx} + u_{yy} = -f(x,y)$ is

$$u(x,y) = S(x,y) * f(x,y) = \iint_{\mathbb{R}^2} S(x-s,y-t)\, f(s,t)\, ds\, dt$$

(Basically the constant is chosen such that $S_{xx} + S_{yy} = -\delta_0$)

Note: Not all solutions are radial!

Example: $u(x,y) = x^2 - y^2$ solves $u_{xx} + u_{yy} = 0$, but isn't radial!