LECTURE 26: LAPLACE EQUATION PROPERTIES

Friday, November 22, 2019 10:31 PM

Today: Study some **unbelievable** properties that solutions of Laplace's equation must satisfy. Prepare to be amazed :)

I- NOTATION

Everything today will be valid in n dimensions, not just 2 dim

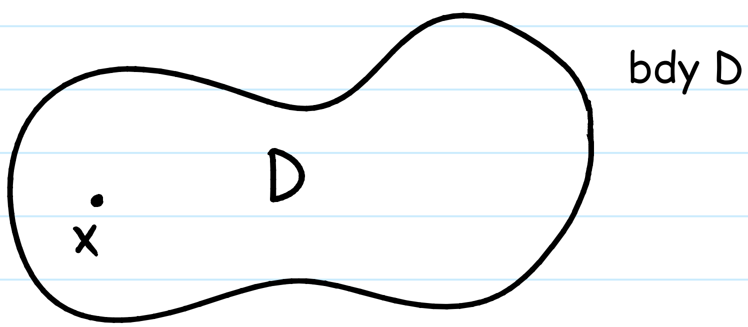
1) $\mathbf{x} = (x_1, \dots, x_n)$ is a point in R^n

2) $\Delta u = u_{x_1 x_1} + \dots + u_{x_n x_n}$

3) Laplace's equation: $\Delta u = 0$

4) D = Some region in R^n with boundary bdy D

(D = open, bounded, connected)



5) $$\int_D f(x)\,dx = \underbrace{\iiint}_{n \text{ times}} f(x)\,dx_1\,dx_2 \dots dx_n$$

II- MEAN-VALUE FORMULA

Let's start with the coolest property :)

Recall: (Math 2B) The average value of f on [a,b] is

$$\frac{\int_a^b f(x)\,dx}{b-a} = \fint_a^b f(x)\,dx$$

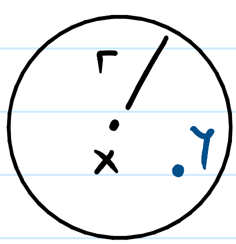
Notice: $b - a$ = Length/Size of $[a,b] = |\,[a,b]\,|$

Definition: The average value of $f(x)$ on D is

$$\fint_D f(x)\,dx = \frac{\int_D f(x)\,dx}{|D|}$$

$|D|$ = Area (or Volume) of D

Notation: $B(x,r)$ = Ball centered at x and radius r



$B(x,r)$

MEAN-VALUE FORMULA

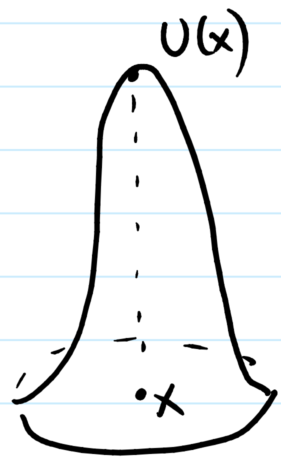
If $\Delta u = 0$ then for every x and $r > 0$ (small)

$$\boxed{\fint_{B(x,r)} u(y)\,dy = u(x)}$$

Interpretation: The average value of u over the ball is just the value at the center!

Example: n = 2 $\quad \dfrac{\iint_{B(x,r)} u(y)\,dy}{\pi r^2} = u(x)$

So solutions to Laplace's equation cannot look like this



In physics, this is sometimes called isotropic (= same from every direction)

Fun consequences:

1) Solutions to $\Delta u = 0$ must be infinitely differentiable (Just like for the heat equation)

Why? $\quad u(x) = \fint_{B(x,r)} u(y)\,dy$ --> 1 level smoother! (because of integral)

(So if u is once differentiable, it becomes twice differentiable, and then thrice differentiable, etc.)

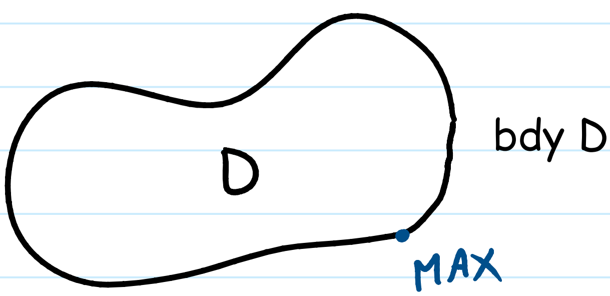
2) **Liouville's Theorem**: If $\Delta u = 0$ on R^n and $|u| \le C$ for some C ("u is bounded"), then u is constant

(So solutions to Laplace's equation MUST blow up somewhere, wow! Also compare to Liouville's theorem in complex analysis)

Most important consequence:

III- THE MAXIMUM PRINCIPLE

Suppose $\Delta u = 0$ in D



Where can you find the maximum/minimum of u?

Theorem: [MAXIMUM PRINCIPLE]

1) [WEAK] $\boxed{\max_{D} u = \max_{\text{bdy } D} u}$

(So the max is attained on bdy D, but in theory could be in D as well)

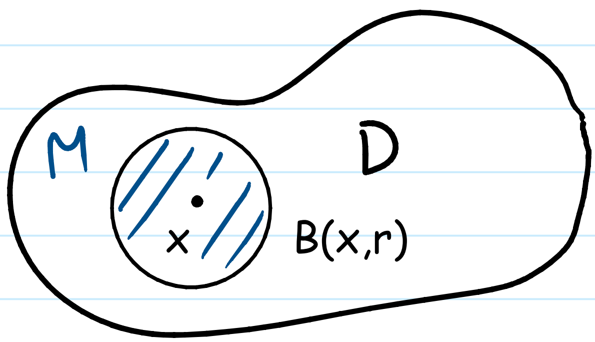
2) [STRONG] u attains its max **ONLY** on bdy D. In fact, if u

attains its max somewhere inside D, then u is constant.

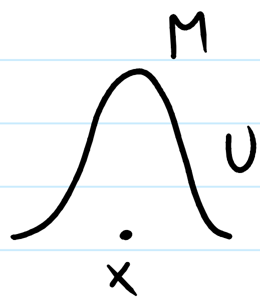
(Of course, everything is true for min, if you replace u by -u)

Why? Only need to show 2) since 2) => 1)

Suppose u attains its maximum (call it M) at some point x in D



(Intuitively: How can u look like this



AND its average value be the value at its center?)

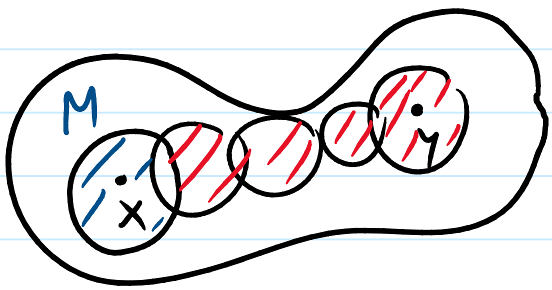
By the mean-value formula (for r small),

$$\fint_{B(x,r)} u(y)\,dy = u(x) = M$$

(Analogy: If you have the highest grade in the class, but also the average grade, then everyone has the same grade as you!)

Therefore, the average value of u is equal to its largest value, so $u \equiv M$ on all of $B(x,r)$

Finally, for any other point y, connect y with x using little balls



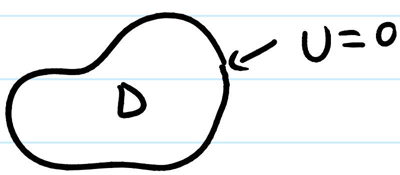
By repeating the proof on each little ball, you get $u \equiv M$ on each ball, and eventually you get $u(y) \equiv M$ and since y was arbitrary, we have $u \equiv M$ on all of D, so u is constant ∎

IV- UNIQUENESS

As usual, we have the following consequences of the Maximum principle:

Consequence 1:

Suppose $\begin{cases} \Delta u = 0 & \text{in } D \\ u = 0 & \text{on bdy } D \end{cases}$



Then $u \equiv 0$

Why? $\max_{D} u = \max_{\text{bdy } D} u = 0$, so $u \le 0$

$\min_{D} u = \min_{\text{bdy } D} u = 0$, so $u \ge 0$

$\Rightarrow u = 0$

Consequence 2: There is at most one solution of

$$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{in bdy } D \end{cases}$$

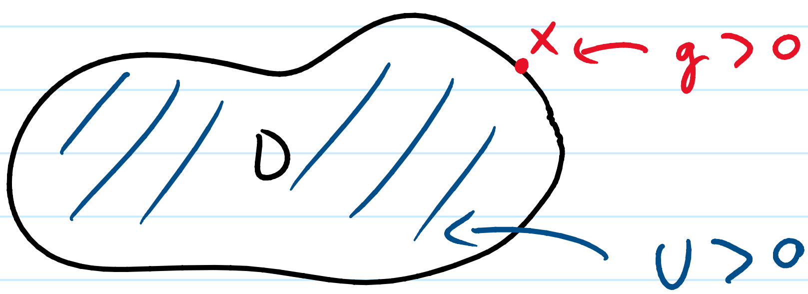
(As usual, suppose u and v are two solutions, let $w = u - v$ ...)

Consequence 3: Positivity

Suppose $\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{in bdy } D \end{cases}$

With $g \geq 0$ but g is positive somewhere (that is, $g \not\equiv 0$)

Then $u > 0$ everywhere in D



Why?

$$\min_D u = \min_{\text{bdy } D} u = \min g \geq 0$$

So $u \geq 0$

Moreover, suppose $u = 0$ somewhere in D, then u attains its minimum (0) inside D, so u is constant, so $u \equiv 0$. But this implies (by continuity) that $g \equiv 0$ =><=

Therefore $u > 0$ everywhere

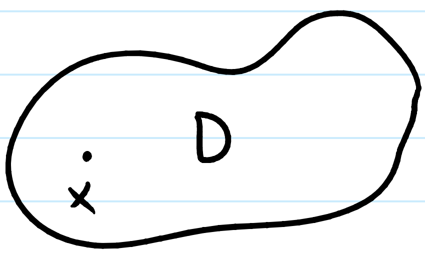
(**Note:** Compare with infinite speed of propagation for the heat equation: If the initial value of u is positive somewhere, then u is positive everywhere)

V- APPLICATIONS

Here is the **real** reason why Laplace's equation is so cool, it has amazing applications!

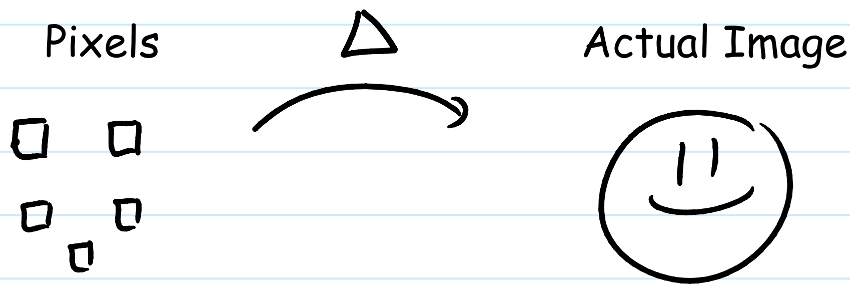
1) PHYSICS

Suppose D is a metal plate (or solid)



$u(x)$ = the temperature of plate x after a loooong time

2) IMAGE PROCESSING



You can use Laplace to convert a pixelated image (= with pixels) to a smooth image. Useful in MRIs or iPhones!

3) COMPLEX ANALYSIS

If $f: C \to C$ is (complex) differentiable, then $Re(f)$ and $Im(f)$ solve Laplace's equation $u_{xx} + u_{yy} = 0$ (Follows from the Cauchy-Riemann equations)

This is another way of obtaining solutions: Take any complex differentiable f and take $Re(f)$ and $Im(f)$

Example: $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{RE} + i\,(\underbrace{2xy}_{IM})$

This says: $x^2 - y^2$ and $2xy$ solve $u_{xx} + u_{yy} = 0$

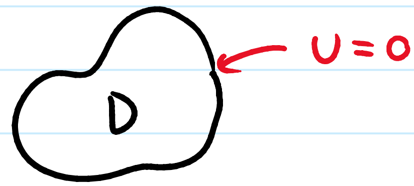
Note: Solutions to Laplace's equation are sometimes called harmonic functions

4) MUSIC

Why harmonic? Comes from music!

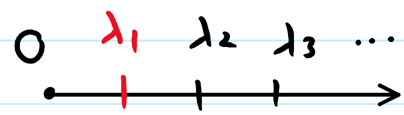
Suppose you have a region D (think the surface of a drum), and consider the following eigenvalue problem: For which λ does the following PDE have a **nonzero** solution?

$$\begin{cases} -\Delta u = \lambda u & \text{in } D \\ u = 0 & \text{on bdy } D \end{cases}$$



Then there exists an infinite sequence of eigenvalues λ_n with

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots$$

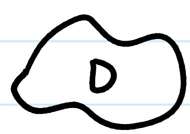


λ_1 is called the principal harmonic (= first sound you hear when you beat a drum), and the other ones are the overtones.

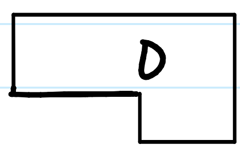
Famous question posed by Mark Kac: "Can you hear the shape of a drum?"

In other words, if I only give you the eigenvalues λ_n, can you figure out what D is?

YES in 2 dimensions if D is smooth



NO if D has corners



NO in higher dimensions; 16-dimensional counterexample