LECTURE 17: EIGENVALUES AND EIGENVECTORS

Sunday, November 3, 2019 3:47 PM

Today: Will cover an important topic called eigenvalues and eigenvectors. It's not clear at first why it's useful, but you'll see in a couple of lectures why it's so useful. By the way, the only reason Google exists is *because* of eigenvalues!

I- MOTIVATION

Example: Consider $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then:

$$\underbrace{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \underbrace{7}_{\lambda}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v}$$

In other words, if you apply A to this specific vector v, you don't just get a random vector, but a multiple of v

$\boxed{\begin{array}{l}\textbf{Definition: If } Av = \lambda v \text{ for some } v \neq 0 \text{, then:} \\ \text{1) } \lambda \text{ is an eigenvalue of } A \\ \text{2) } v \text{ is an eigenvector of } A \text{ (corresponding to } \lambda\text{)}\end{array}}$

Example: In the above example:

$\lambda = 7$ is an eigenvalue of A and
$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 7$

Example: $A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$ $\quad v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$Av = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \lambda v$$

$\lambda = 2$ is an eigenvalue and $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is an eigenvector

BUT

$$\begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(for any } \lambda\text{)}$$

$A v \neq \lambda v$, so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not an eigenvector of A

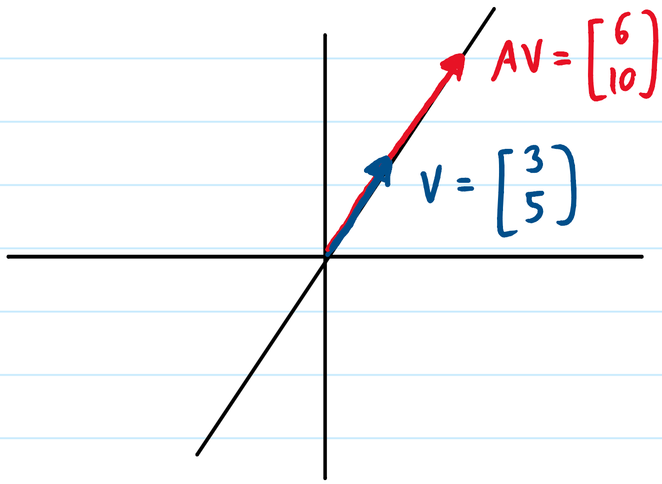
Note: Eigenvectors are really special vectors

Geometric Interpretation:

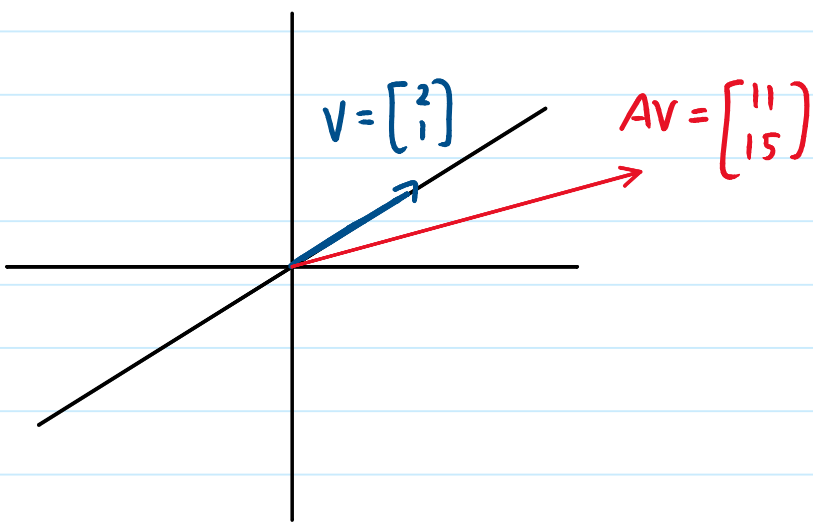
$Av = \lambda v \Rightarrow$ Both Av and v lie on the same line

Picture:

<u>Eigenvector</u>



Not eigenvector



So eigenvectors are very special! In physics, eigenvectors usually indicate a resonance effect, as in the first picture above

II- FINDING EIGENVALUES

Example: Find all the eigenvalues of $A = \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$

Motivation: Suppose $Av = \lambda v$ ($v \neq 0$)

$Av = \lambda v$
<=> $\lambda v - Av = 0$
<=> $(\lambda I - A)v = 0$
(have to put I because you can't subtract a number from a matrix)
<=> $v \neq 0$ is in $\mathrm{Nul}(\lambda I - A)$
<=> $(\lambda I - A)$ is not invertible ("$Ax = 0$" has a nonzero solution)
<=> $\det(\lambda I - A) = 0$

$$\boxed{\text{FACT: } \lambda \text{ is an eigenvalue of } A \Leftrightarrow \det(\lambda I - A) = 0}$$

Note:

1) $\det(\lambda I - A)$ is called the characteristic equation of A, helps us find λ
2) Mnemonic: $\det(\lambda I - A)$ sounds like λ IAR
3) Totally fine to use $A - \lambda I$, but this is better because you'll make fewer sign mistakes.

Here:

$$A = \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{vmatrix} \lambda & -6 \\ 1 & \lambda - 5 \end{vmatrix}$$

(BASICALLY: Put λ on diagonal and put minus signs on all the entries of A)

$$= \lambda(\lambda-5) + 6$$
$$= \lambda^2 - 5\lambda + 6 \qquad \text{(Characteristic equation)}$$
$$=(\lambda-2)(\lambda-3)$$
$$= 0$$

=> $\lambda = 2$ and $\lambda = 3$

Example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\det(\lambda I-A) = \begin{vmatrix} \lambda-1 & -6 \\ -5 & \lambda-2 \end{vmatrix}$$

(again, put λ on diagonal, and minus everything)

$$= (\lambda-1)(\lambda-2) - 30$$
$$= \lambda^2 - 3\lambda + 2 - 30$$
$$= \lambda^2 - 3\lambda - 28$$
$$= (\lambda-7)(\lambda+4)$$
$$= 0$$

=> $\lambda = 7$ and $\lambda = -4$

III- FINDING EIGENVECTORS

Example: "Find the eigenvectors" of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Back to: $Av = \lambda v \iff (\lambda I - A)v = 0 \iff$ $\boxed{v \text{ is in } \mathrm{Nul}(\lambda I - A)}$

STRATEGY: For each λ you found, find Nul(λI - A)

Found: $\lambda = 7$ and $\lambda = -4$

<u>$\lambda = 7$</u>

$$\text{Nul}(7I - A) = \text{Nul}\begin{bmatrix} 7-1 & -6 \\ -5 & 7-2 \end{bmatrix}$$ (plug in the above, but with $\lambda = 7$)

$$= \text{Nul}\begin{bmatrix} 6 & -6 \\ -5 & 5 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$ (MUST use RREF)

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad x - y = 0 \Rightarrow x = y$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $\text{Nul}(7I - A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for $\underbrace{\text{the set of eigenvectors corresponding to } \lambda = 7}_{\text{Eigenspace of } \lambda = 7}$

$$\underbrace{\text{Eigenspace of } \lambda = 7}_{E_7}$$

Note: Why do we get so many eigenvectors? Because if v is an eigenvector, so is any multiple of v ! (anything on the same line as v is still an eigenvector)

<u>$\lambda = -4$</u>

$$\text{Nul}(-4I - A) = \text{Nul}\begin{bmatrix} -4-1 & -6 \\ -5 & -4-2 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} -5 & -6 \\ 0 & 0 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & 6/5 \\ 0 & 0 \end{bmatrix} \quad \text{(RREF)}$$

$(\Rightarrow x + 6/5\, y = 0 \Rightarrow x = -\,6/5\, y)$

$$= \text{Span}\left\{\begin{bmatrix} -6/5 \\ 1 \end{bmatrix}\right\}$$

$$= \text{Span}\left\{\begin{bmatrix} -6 \\ 5 \end{bmatrix}\right\} \quad \text{(OK to scale eigenspaces)}$$

$\left\{\begin{bmatrix} -6 \\ 5 \end{bmatrix}\right\}$ is a basis for E_{-4}

WARNING: You should NEVER EVER GET Nul = {**0**}

(because an eigenvector is **precisely** a vector that makes this nonzero)