# LECTURE 18: DIAGONALIZATION

Monday, November 4, 2019 3:12 PM

## I- THE 3x3 CASE (section 5.2)

Let's continue our eigenvalue extravaganza! The good news is that for bigger matrices, the process of finding eigenvalues/eigenvectors is exactly the same; just the algebra is messier!

**Example:** Find the eigenvalues of

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 2 & -2 \\ -4 & \lambda-3 & 4 \\ 0 & 2 & \lambda-1 \end{vmatrix}$$

$$= 0\,(\ldots) \quad - 2 \begin{vmatrix} \lambda+1 & -2 \\ -4 & 4 \end{vmatrix} + (\lambda-1) \begin{vmatrix} \lambda+1 & 2 \\ -4 & \lambda-3 \end{vmatrix}$$

$$= -2\,[4(\lambda+1)-8] + (\lambda - 1)[(\lambda + 1)(\lambda - 3)+8]$$

$$= -2[4\lambda + 4 - 8] + (\lambda-1)[\lambda^2 - 2\lambda -3 + 8]$$

$$= -2\,(4\lambda - 4) + (\lambda - 1)(\lambda^2 - 2\lambda + 5)$$

$$= -2(4)(\lambda - 1) + (\lambda - 1)(\lambda^2 - 2\lambda + 5)$$

$$= (\lambda - 1)(-8 + \lambda^2 - 2\lambda + 5)$$

$$= (\lambda - 1)(\lambda^2 - 2\lambda - 3)$$

$$= (\lambda - 1)(\lambda - 3)(\lambda + 1)$$

$$= 0$$

=> $\lambda = 1, \lambda = 3, \lambda = -1$

(In all those problems, there's always a common factor like $(\lambda - 1)$ here that's standing out)

**Note:** You can then find the eigenvectors just as before

**Example:** $\underline{\lambda = 1}$

$$\text{Nul}(1I - A) = \text{Nul}\begin{bmatrix} 1+1 & 2 & -2 \\ -4 & 1-3 & 4 \\ 0 & 2 & 1-1 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 2 & 2 & -2 \\ -4 & -2 & 4 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \dots$$

$$= \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(RREF)}$$

$$= \dots$$

$$= \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

## II- DIAGONALIZATION (section 5.3)

Today's lecture is all about diagonal matrices

**Ex:** $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Today's Goal: Turn a matrix A into a diagonal matrix D

**Ex:** $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ can be turned into $D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$

(Hard) (Easier)

What does "can be turned into" mean?

**Definition:** A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$\boxed{A = PD(P^{-1})}$$

**Note:** If you illegally cancel out P and $P^{-1}$ in $A = \cancel{P}D\cancel{P^{-1}}$ you get A = D, so A is like D (also see analogy below)

**Example:**

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ (from last time)

Find D diagonal and P invertible such that $A = PDP^{-1}$

Last time: Found

$\lambda = 7$ (eigenvalue) with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -4$ with eigenvector $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$

Answer:

$D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$ (matrix of eigenvalues)

$P = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$ (matrix of corresponding eigenvectors)

$(\lambda = 7) \ (\lambda = -4)$

<u>WHY DOES THIS WORK?</u>

Notice:

$$AP = A \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$$

$$= \left[ A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad A\begin{bmatrix} -6 \\ 5 \end{bmatrix} \right]$$

$$= \left[ 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -4\begin{bmatrix} -6 \\ 5 \end{bmatrix} \right]$$ (def. of eigenvector)

$$\begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{bmatrix} \quad \text{(def. of eigenvector)}$$

$$= \begin{bmatrix} 7 & 24 \\ 7 & -20 \end{bmatrix}$$

But also $PD = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 24 \\ 7 & -20 \end{bmatrix}$

$$= \begin{bmatrix} 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} & -4\begin{bmatrix} -6 \\ 5 \end{bmatrix} \end{bmatrix}$$

So $AP = PD \Rightarrow A\cancel{P}(\cancel{P}^{-1}) = PD(P^{-1}) \Rightarrow A = PDP^{-1}$ !!!

Upshot: This process (called diagonalization) just boils down to finding eigenvalues and eigenvectors of A

**Example:** (Good exam problem)

Find D diagonal and P invertible such that $A = PDP^{-1}$

Where $A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$

Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda-7 & 3 \\ -10 & \lambda+4 \end{vmatrix}$$

$$= (\lambda-7)(\lambda+4) + 30$$
$$= \lambda^2 - 3\lambda - 28 + 30$$

$$= \lambda^2 - 3\lambda + 2$$
$$= (\lambda-1)(\lambda-2)$$
$$= 0$$

=> $\lambda = 1$ and $\lambda = 2$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

<u>Eigenvectors:</u>

<u>$\lambda = 1$</u> $\quad \text{Nul}(1I - A) = \text{Nul}\begin{bmatrix} 1-7 & 3 \\ -10 & 1+4 \end{bmatrix}$

$$= \text{Nul}\begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \text{ (RREF)}$$

$$= \ldots$$
$$= \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

$\underline{\lambda = 2}$ $\quad \text{Nul}(2I - A) = \text{Nul}\begin{bmatrix} 2-7 & 3 \\ -10 & 2+2 \end{bmatrix}$

$$= \text{Nul}\begin{bmatrix} -5 & 3 \\ -10 & 4 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} -5 & 3 \\ -5 & 3 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix}$$

$$= \ldots$$

$$= \text{Span}\left\{\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

$\nearrow \qquad \uparrow$

$\lambda = 1 \quad \lambda = 2$

**Example:** (Even better exam problem)

Same but

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

<u>Eigenvalues</u>:

$|\lambda I - A| = \ldots = (\lambda+1)(\lambda-3)^2$

$\lambda = -1, 3$

Eigenvectors:

$\underline{\lambda = -1}$ ---> $\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

$\underline{\lambda = 3}$ ----> $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(Have to repeat $\lambda = 3$ twice because 2 eigenvectors for $\lambda = 3$; and also had $(\lambda - 3)^2$ in the characteristic equation)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$\lambda = -1 \quad \lambda = 3$

Note: The order of the eigenvalues/vectors doesn't matter, as long as the eigenvector goes with the correct eigenvalue.

## III- ANALOGY

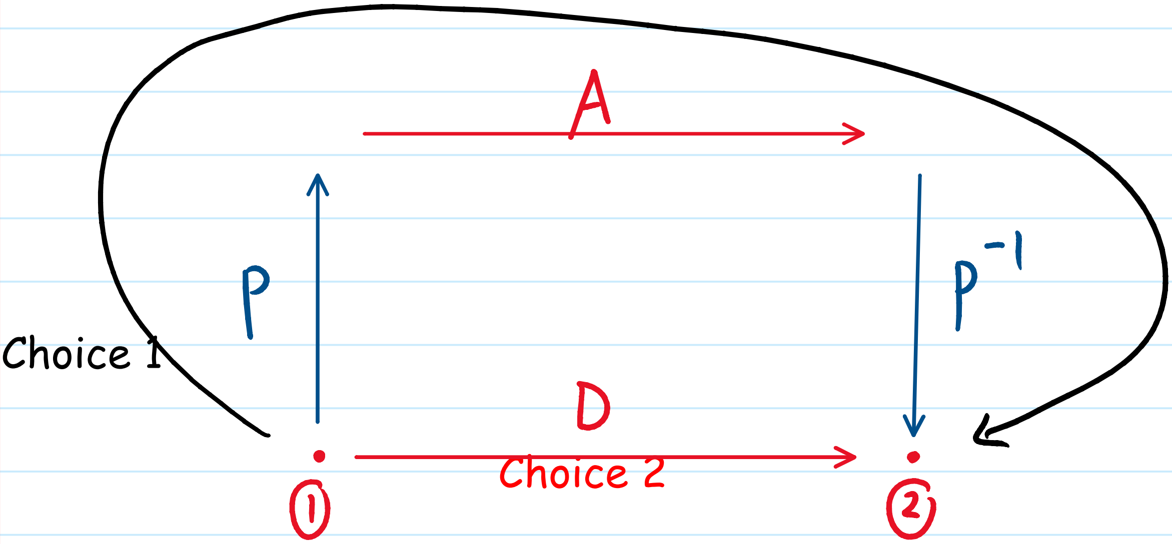
You may wonder: Why does $A = PDP^{-1}$ imply that $A$ is like $D$?
And also, does $PDP^{-1}$ appear in nature?

Notice:

$$\begin{aligned} A = PDP^{-1} &\Rightarrow \quad P^{-1} A P = P^{-1} P D P^{-1} P \\ &\Rightarrow \quad P^{-1} A P = D \\ &\Rightarrow D = P^{-1}AP \end{aligned}$$

What does this mean?

Suppose you want to fly from City 1 to City 2



then you have two choices:

Choice 1: Take flight $P$, *then* take flight $A$, *then* take flight $P^{-1}$.
This *whole* thing is called $P^{-1}AP$ (read this from right to left, like Arabic/Farsi)

Choice 2: Take a direct flight D that brings you from 1 to 2

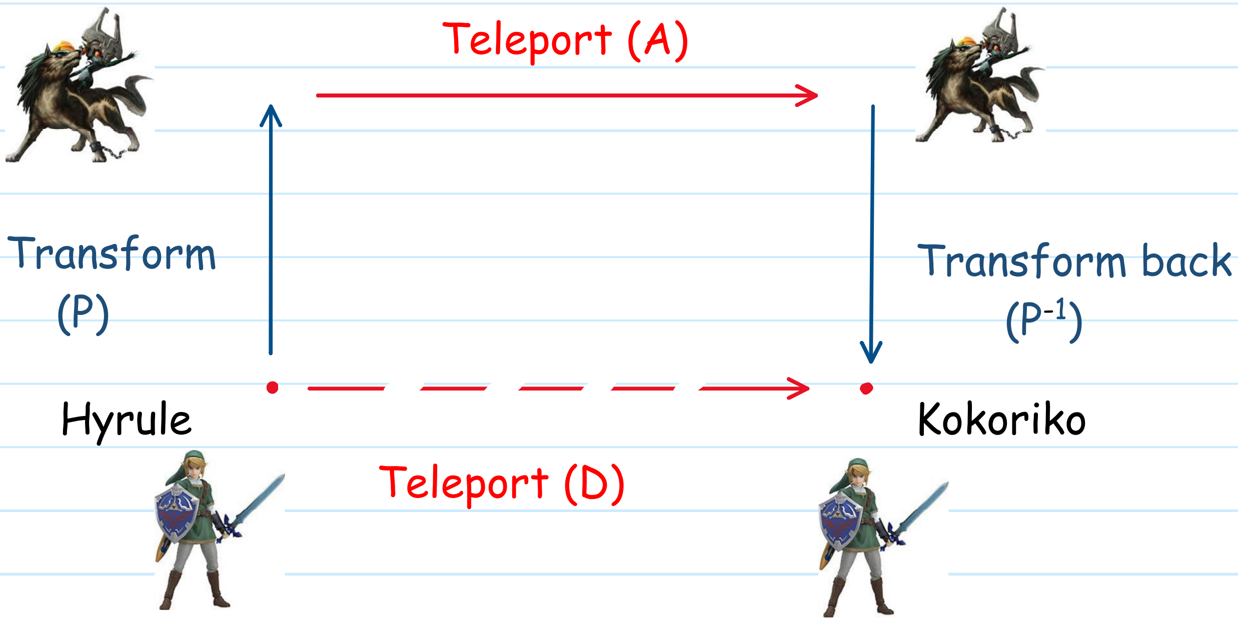
$P^{-1}AP = D$ says that the two choices are the same.

**IN PARTICULAR:** If you ignore the vertical P and $P^{-1}$ lines (which sort of cancel out anyway) then A is "like" D

**ANALOGY:** In "Legend of Zelda - Twilight Princess," there's this really annoying process:

Suppose you want to teleport yourself from, say, Hyrule to Kokoriko, then: you first transform yourself into a wolf (P), then teleport yourself as a wolf (A), then transform yourself back ($P^{-1}$). What $P^{-1}AP = D$ is saying is that this whole transform-teleport-transform process ($P^{-1}AP$) is the same as just teleporting yourself as Link (D).

In particular, if you ignore the Link-Wolf transformation (= ignore P and $P^{-1}$), then teleporting yourself as a wolf (A) is really like teleporting yourself as Link (D)



This is why $A = PDP^{-1}$ really means A is "like" D