LECTURE 19: DIAGONALIZABILITY

Tuesday, November 5, 2019 5:06 PM

Let's continue our diagonalization adventure!

I- DIAGONALIZABILITY

Definition: A is diagonalizable if there are D and P such that

$$A = PDP^{-1}$$

Example: Is A diagonalizable?

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Last time:

$\underline{\lambda = -1}$ ---> $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\underline{\lambda = 3}$ ----> $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

YES (we literally found D and P such that this works)

Note: MANY choices of D and P are acceptable, as long as the eigenvector goes with the correct eigenvalue!

Question: Is every matrix diagonalizable?

Sadly the answer is NO

Example: (EXCELLENT counterexample)

Same, but

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1) Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 = 0$$

=> $\lambda = 1$ (repeated twice)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2) Eigenvectors:

$\underline{\lambda = 1}$ $\quad \text{Nul}(I - A) = \text{Nul}\begin{bmatrix} 1-1 & -1 \\ 0 & 1-1 \end{bmatrix}$

$$= \text{Nul}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Not square!

$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$? Not invertible!

IN FACT: P does not exist! Here A is **NOT** diagonalizable!

Which raises the question:

WHEN is A diagonalizable?

IMPORTANT FACT:

A (n x n) is diagonalizable <=> A has n (LI) eigenvectors

Here: A has only 1 (LI) eigenvector, so NO

Ex: $A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$ has 2 LI eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

So yes

In fact, that's how we constructed P:

$P = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ (just put the eigenvectors together)

Useful test:

IF A (nxn) has n distinct eigenvalues, then A is diagonalizable

Example: $A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -3 \\ 0 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = 2, \lambda = 4$

Two different eigenvalues, so YES

(**Note**: For an upper-triangular matrix, the eigenvalues are on the diagonal)

BUT: A diagonalizable =/=> A has n distinct eigenvalues!

A can be diagonalizable even with just 1 or 2 eigenvalues!

Example:

$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

Has only two eigenvalues $\lambda = 3$ and $\lambda = -1$, but was still diagonalizable (see above) because it had 3 (LI) eigenvectors!

Example:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Is diagonalizable (it's diagonal!) but only has eigenvalue $\lambda = 1$

Moral: To check for diagonalizability, really have to look at the

eigenvectors! (unless it has n distinct eigenvalues)

Example: Suppose A is 7x7, has eigenvalues 3 and 4 with $\dim(E_3) = 5$ and $\dim(E_4) = 2$. Is A diagonalizable?

YES, this basically says have 5 + 2 = 7 (LI) eigenvectors

Sidenote:

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A is invertible <=> 0 is not an eigenvalue of A

(but in general, diagonalizability has nothing to do with invertibility)

II- ANALOGY

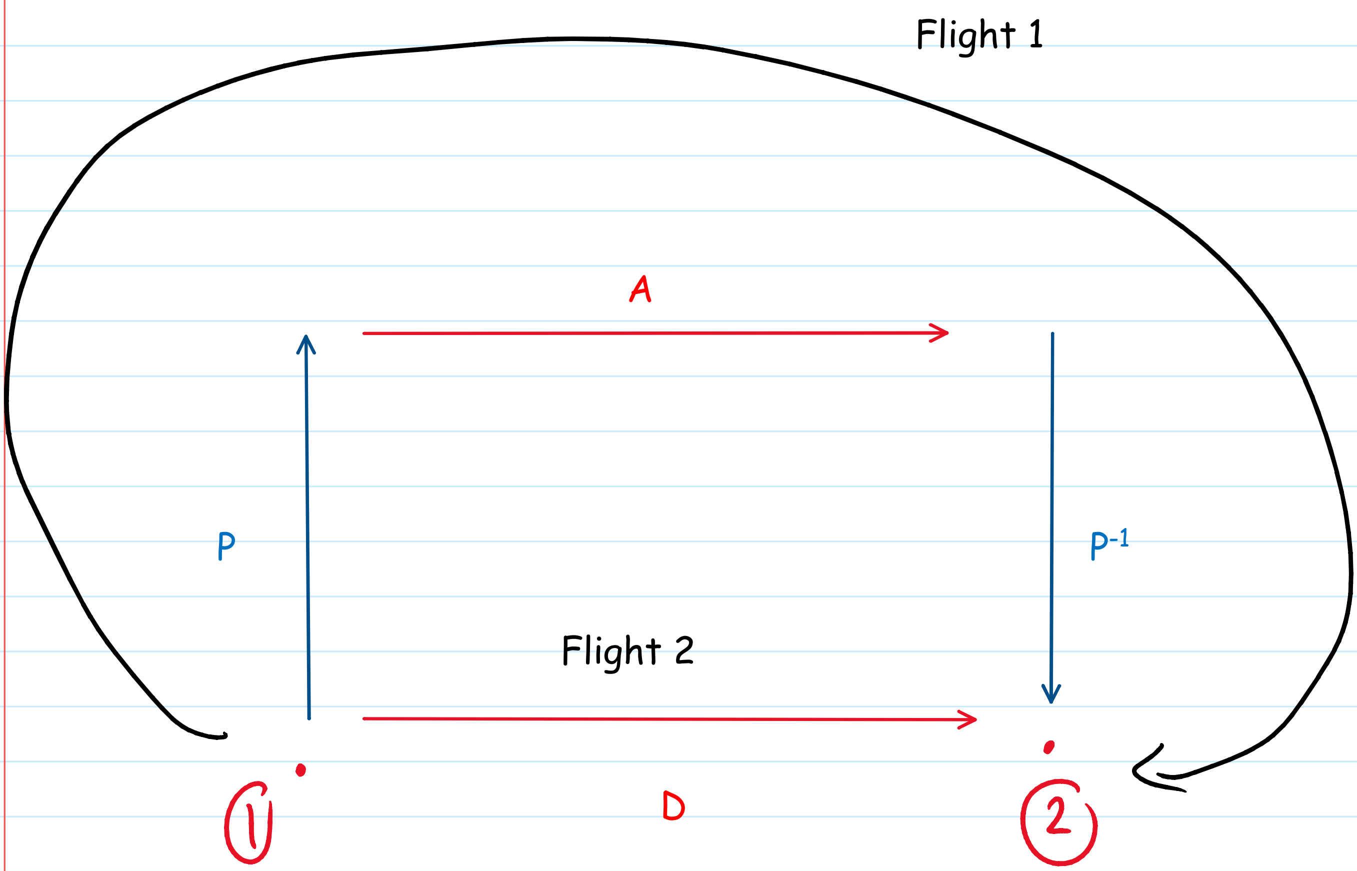
You may wonder: Why does $A = PDP^{-1}$ imply that A is like D? And also, does PDP^{-1} appear in nature?

Notice:

$$A = PDP^{-1} \Rightarrow P^{-1} A P = P^{-1} P D P^{-1} P$$
$$\Rightarrow P^{-1} A P = D$$
$$\Rightarrow D = P^{-1}AP$$

What does this mean?

Suppose you want to fly from City 1 to City 2



Then you have 2 choices of flights:

Flight 1: Take a flight P, then a flight A, then a flight P^{-1}
This whole journey is called $P^{-1} A P$ (read from right to left)

Flight 2: Take a direct flight D

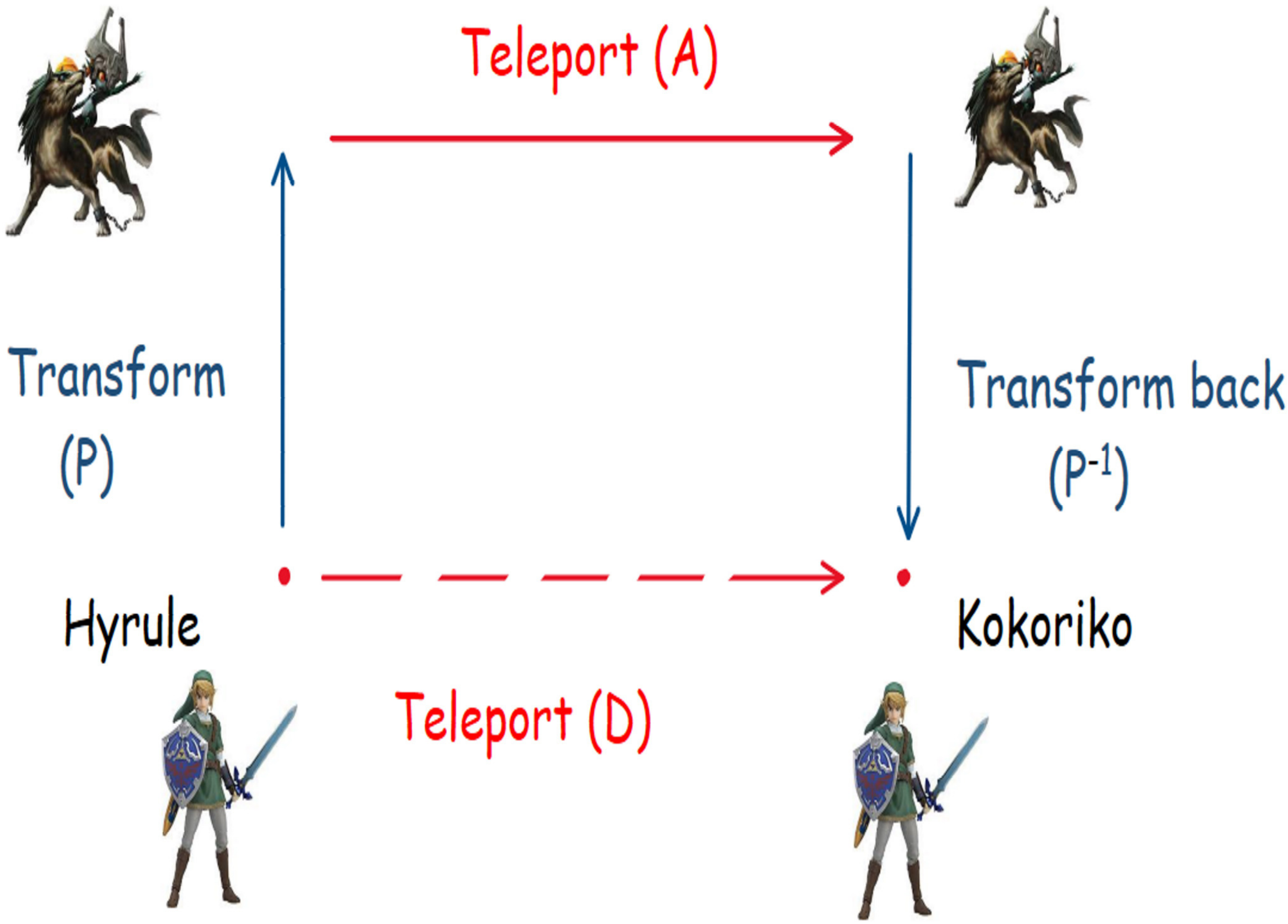
$P^{-1} AP = D$ says that the two flights bring you to the same destination

Moreover, if you ignore the vertical lines P and P^{-1}, then you get $A = D$

ANALOGY: In "Legend of Zelda - Twilight Princess," there's this really annoying process:

Suppose you want to teleport yourself from, say, Hyrule to Kokoriko, then: You first transform yourself into a wolf (P), then you teleport yourself as a wolf (A), and then you transform yourself back (P^{-1}).

This whole process is $P^{-1} A P$



What $A = PDP^{-1}$ is saying is that this whole process is the same as just teleporting yourself as Link (which is just D)

In particular, if you ignore the Link to Wolf transformation, then indeed teleporting yourself as a wolf (= A) is like teleporting yourself as Link (= D), so A is "like" D

In fact, you might even say... A is SIMILAR to D!

III- SIMILARITY

In fact, diagonalizability is part of a more general concept

Definition: A is similar to B (A ~ B) if there is an invertible matrix P such that

$$A = PBP^{-1}$$

(= A is "like" B)

Example: If A is diagonalizable, then $A = PDP^{-1}$, so A is similar to a diagonal matrix D (= A is like a diagonal matrix)

Similar matrices indeed share similar properties

Example: Show that A ~ B => det(A) = det(B)

$A = PBP^{-1}$ for some P

$$\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1})$$
$$= \det(P)\det(B)(1/\det(P))$$
$$= \det(B)$$

Example: Show that A ~ B => $A^2 \sim B^2$

Suppose $A = PBP^{-1}$

Then $A^2 = AA = (PBP^{-1})(PBP^{-1}) = PBBP^{-1} = P(B^2) P^{-1}$

Hence $A^2 \sim B^2$

Example: Show that $A \sim B \Rightarrow B \sim A$

$A = PBP^{-1}$

Solve for B:

$P^{-1} A = P^{-1} PBP^{-1} = BP^{-1}$

$(P^{-1} A) P = B P^{-1} P = B$

$B = P^{-1} A P = (P^{-1}) A (P^{-1})^{-1} = Q A Q^{-1}$ for $Q = P^{-1}$

(Point: Doesn't have to be the same P !)

Example: Show that if A is similar to B and B is diagonalizable, then A is diagonalizable

Know: $A = PBP^{-1}$ and $B = QDQ^{-1}$

Then $A = PBP^{-1} = P(QDQ^{-1})P^{-1} = (PQ)D(PQ)^{-1} = RDR^{-1}$ for $R = PQ$

So $A = (\text{Blah})\, D\, (\text{Blah})^{-1}$ with D diagonal

Hence A is diagonalizable.