LECTURE 20: B-MATRICES

Friday, November 8, 2019 3:21 PM

Today: Let's start with a topic that has nothing to do with eigenvectors!

I- RECAP: MATRIX OF A LINEAR TRANSFORMATION

Example: Find the ($\mathcal{B}$-) matrix of $T\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$

where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (Basis of R^2)

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are labelled $\mathcal{B}$)

Answer: $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ "$\mathcal{B}$-matrix of A"

General Strategy: To find the $\mathcal{B}$-matrix of A

1. For every **b** in $\mathcal{B}$, calculate A**b**
2. Write the result in terms of $\mathcal{B}$

II- $\mathcal{B}$-MATRIX

Example: Find the $\mathcal{B}$-matrix of $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ where

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \text{(given)}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underline{\underline{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underline{\underline{0}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

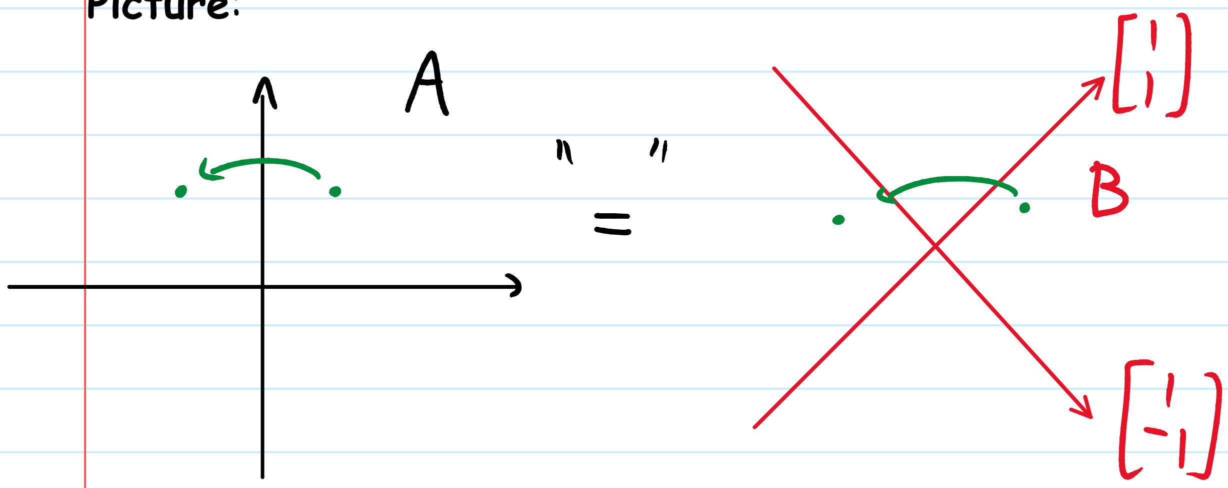
$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underline{\underline{1}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underline{\underline{1}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Interpretation:

B "=" A, but in the new coordinates $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Picture:



Example: Find the $\mathscr{B}$-matrix of $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$

$$\mathscr{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Solve: $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right] \Rightarrow a = 1,\ b = -2$$

Says:

$$A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

SAME MATRIX! (call it P)

Answer: $B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Faster Way: Since you're doing the same row-reduction anyway,

$$[P \mid AP] = \left[\begin{array}{cc|cc} 3 & -1 & 5 & 5 \\ 2 & 1 & 0 & 5 \end{array}\right] \text{----}> \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] = [I|B]$$

$$B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 2 & 7 \end{bmatrix}$$

$$[P|AP] = \left[\begin{array}{cc|cc} 1 & 5 & 2 & 9 \\ 1 & 4 & 2 & 7 \end{array}\right] \text{----}> \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right] = [I|B]$$

$$B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Notice: To construct B, you first calculate AP and then do P^{-1} of that, so in fact:

$B = P^{-1} AP \Rightarrow A = PBP^{-1}$

=> **Fact:** A is always similar to its B-matrix!

Which leads me to the next topic...

III- B-MATRIX AND EIGENVECTORS

What does that have to do with eigenvectors?

Example: $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -2 \end{bmatrix} \dashrightarrow \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} \dashrightarrow \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

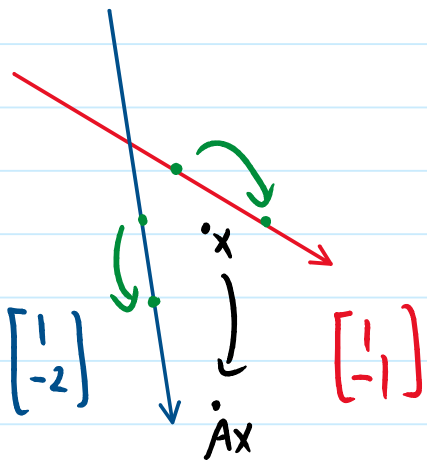
$B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ DIAGONAL!!!

Coincidence??? I think not!

Notice: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors of A !!!

=> **FACT:** If $\mathcal{B}$ is a basis of eigenvectors of A, then B is diagonal!

Interpretation:



If your new axes are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ then A "=" $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

That is, A stretches vectors on the $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ axis by 5

And stretches vectors on the $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ axis by 3

And for any other vector, A does a mix of the two

(This gives us a **complete** characterization of what A does geometrically! WOW!)

Example: Find a basis $\mathcal{B}$ for which B is diagonal, where

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda-5 & -4 \\ -4 & \lambda-5 \end{vmatrix} = (\lambda-5)^2 - 4^2 = (\lambda-5-4)(\lambda-5+4) = (l-9)(l-1) = 0$$

$\Rightarrow \lambda = 1, 9$

Eigenvectors:

$\underline{\lambda = 1}$ ---> $\text{Nul}\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\underline{\lambda = 9}$ ---> $\text{Nul}\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

III- APPLICATION 1: $\sqrt{A}$

This ends all the dry theory from Chapter 5, and for the next couple of lectures, we'll just do fun applications, so you can really see **WHY** diagonalization is so useful!

Note: All the applications are just based on the following observation:

If $A = PDP^{-1}$, then

$A^2 = AA = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, and generally

$A^n = P D^n P^{-1}$

Moreover, if, say, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ then $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$

Point: D is easy to calculate, and hence A is easy to calculate!

Example: Find $\sqrt{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}$

$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

Find D and P:

$\lambda = 1 \dashrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \lambda = 9 \dashrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$ $\qquad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$A = PDP^{-1}$

$\sqrt{A} = P\sqrt{D}\,P^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{9} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

And in fact:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = A \text{ !!! (WOW)}$$

Note: In the same way, can define e^A, $\sin(A)$, and even A^B (see YouTube)

More applications next time!