LECTURE 21: APPLICATIONS OF DIAGONALIZATION

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Today: 3 AMAZING applications of diagonalization, so that you can finally understand why this chapter is so useful!

I- PRELUDE

Note: All the applications are just based on the following observation:

If $A = PDP^{-1}$, then

$A^2 = AA = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, and generally

$A^n = P D^n P^{-1}$

Moreover, if, say, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ then $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$

Point: D is easy to calculate, and hence A is easy to calculate!

Note: Using this idea, can also calculate $\sqrt{A}$, e^A, $\cos(A)$ (see YouTube)

II- APP 1: POKÉMON BATTLE

Welcome to Pokémon stadium, where we're about to witness a fierce battle between Charmander and Pikachu!

(The way Pokémon works is that there are several rounds, and at each round both Pokémon attack each other, and they win/lose HP. The game is over if one of them has 0 HP)

C_n = HP of Charmander after round n
P_n = HP of Pikachu after round n

Initially: $C_0 = 100$
$P_0 = 46$

And:
$$\begin{cases} C_{n+1} = 16\,C_n - 35\,P_n \\ P_{n+1} = 6\,C_n - 13\,P_n \end{cases}$$

(After each round, Charmander heals himself to gain 16Cn but also gets attacked by Pickachu and loses 35Pn
Pikachu steals HP from Charmander and gains $6C_n$ but electrocutes himself and loses $13P_n$)

Goal: 1) Find a formula for P_n and C_n
2) Who loses first?

STEP 1: Matrix Form

$$\begin{bmatrix} C_{n+1} \\ P_{n+1} \end{bmatrix} = \begin{bmatrix} 16 & -35 \\ 6 & -13 \end{bmatrix} \begin{bmatrix} C_n \\ P_n \end{bmatrix}$$

Next A Now

STEP 2: Diagonalize A

$A = PDP^{-1}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

STEP 3:

$$A^n = P\,D^n P^{-1}$$
$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1}$$
$$= \ldots$$
$$= \begin{bmatrix} -14 + 15(2^n) & 35 - 35(2^n) \\ -6 + 6(2^n) & 15 - 14(2^n) \end{bmatrix} \quad (*)$$

STEP 4:

$$\begin{bmatrix} C_1 \\ P_1 \end{bmatrix} = A \begin{bmatrix} C_0 \\ P_0 \end{bmatrix}$$

$$\begin{bmatrix} C_2 \\ P_2 \end{bmatrix} = A \begin{bmatrix} C_1 \\ P_1 \end{bmatrix} = A \; A \begin{bmatrix} C_0 \\ P_0 \end{bmatrix} = A^2 \begin{bmatrix} C_0 \\ P_0 \end{bmatrix}$$

$$\begin{bmatrix} C_n \\ P_n \end{bmatrix} = A^n \begin{bmatrix} C_0 \\ P_0 \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} 100 \\ 46 \end{bmatrix} = \begin{bmatrix} 210 - 110\,(2^n) \\ 90 - 44\,(2^n) \end{bmatrix}$$

Ex: $\begin{bmatrix} C_1 \\ P_1 \end{bmatrix} = \begin{bmatrix} 210 - 110(2) \\ 90 - 44(2) \end{bmatrix} = \begin{bmatrix} -10 \\ 2 \end{bmatrix}$ => Charmander Loses!

III- APP 2: PRESS F FOR FIBONACCI

(Consider the following "fun" game: Start with 0 and 1, and the next number is the sum of the previous 2)

0, 1, 1, 2, 3, 5 , 8, 13, ...

(Fibonacci used it to count... bunnies)

$$\begin{cases} F_{n+1} = F_n + F_{n-1} \quad (\Rightarrow \text{Future} = \text{Present} + \text{Past}) \\ F_0 = 0 \\ F_1 = 1 \end{cases}$$

Goal: Find a formula for F_n

Trick: Add the following "trivial" equation

$$\begin{cases} F_{n+1} = F_n + F_{n-1} \\ F_n = \quad F_n \end{cases}$$

(**Why?** That way we can turn it into matrix form)

STEP 1: Matrix Form

$$\underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{\text{Next}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}}_{\text{Now}}$$

STEP 2: Diagonalize A

$A = PDP^{-1}$ $\quad D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $\quad P = \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}$

$a = \frac{1+\sqrt{5}}{2} = \phi$ (Golden Ratio), $b = \frac{1-\sqrt{5}}{2} = 1-a$

STEP 3:

$$A^n = PD^n P^{-1} = \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \ldots$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} a^{n+1} - b^{n+1} & a(b^{n+1}) - b(a^{n+1}) \\ a^n - b^n & a(b^n) - b(a^n) \end{bmatrix} \quad (*)$$

STEP 4:

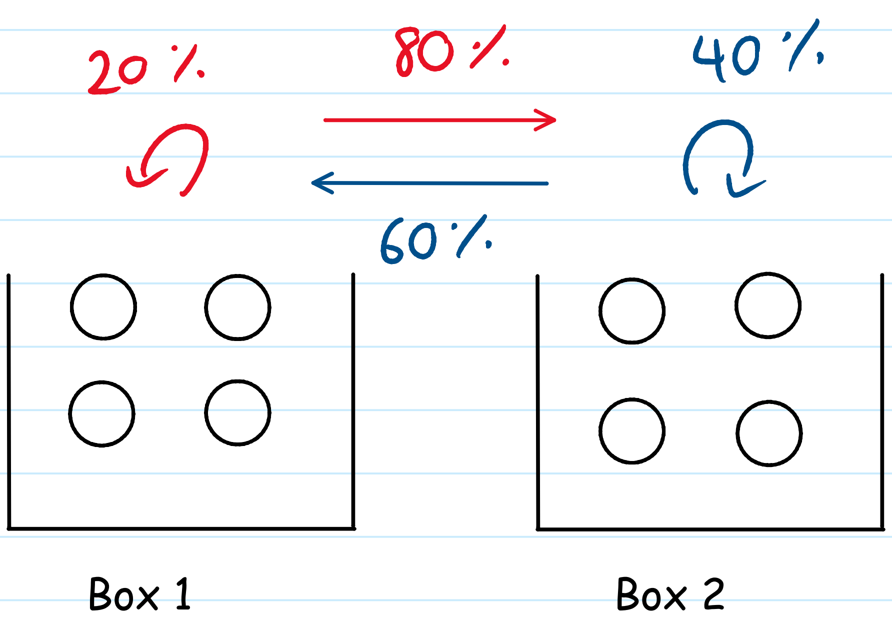
$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (*) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & (a^{n+1} - b^{n+1}) \\ 1/\sqrt{5} & (a^n - b^n) \end{bmatrix}$$

Answer:

$$F_n = 1/\sqrt{5}\,(a^n - b^n) = 1/\sqrt{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N\right)$$

WOW, who would have thought???
At each step, the square roots magically cancel out to give you the sequence 0, 1, 1, 2, 3, 5, 8, ...

IV- APP 3: TAKE IT TO THE LIMIT!



(**Setting:** Start with 2 Boxes with equal number of balls
At each step, take 80 % of the balls from 1 and put it in 2, and at the same time, take 60 % of the balls from 2 and put it in 1)

X_n = # of balls in Box 1 after n steps
Y_n = # of balls in Box 2 after n steps

Initially:

$X_0 = 0.5$
$Y_0 = 0.5$

(Means: 50 % of balls are in 1 and 50 % in 2)

$$\begin{cases} X_{n+1} = 0.2\, X_n + 0.6\, Y_n \\ Y_{n+1} = 0.8\, X_n + 0.4\, Y_n \end{cases}$$

(**Why?** The only way to be in 1 at the next round is either if you were in 1 to begin with (20 % chance) or if you were transferred from 2 (60 % chance))

Question: What happens in the long-run? What are X_∞ and Y_∞ ?

STEP 1: Matrix Form

$$\begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}}_{A} \begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$

STEP 2:

$A = PDP^{-1}$, $D = \begin{bmatrix} -0.4 & 0 \\ 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$

STEP 3:

$A^n = P\, D^n\, P^{-1}$

$$A^\infty = P\, D^\infty\, P^{-1}$$
$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} (-0.4)^\infty & 0 \\ 0 & 1^\infty \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (1/7) \begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix}$$

STEP 4:

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = A^n \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$\begin{bmatrix} X_\infty \\ Y_\infty \end{bmatrix} = A^\infty \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 3/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

(So in the end, about 3/7 of the balls will be in 1 and 4/7 will be in 2)

AMAZING FACT: This does NOT depend on X_0 and Y_0

Why? Suppose $X_0 = a$, then $Y_0 = 1-a$

(Ex: If $X_0 = 0.25$, then $Y_0 = 0.75 = 1-0.25$)

Then

$$\begin{bmatrix} X_\infty \\ Y_\infty \end{bmatrix} = A^\infty \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 3/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix} \begin{bmatrix} a \\ 1-a \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

Note: Will see a **SICK** application of this next time!