LECTURE 22: DOT PRODUCT AND LENGTHS

One last application of diagonalization! And this application affects all of us, because let me show you how Google works!

I- APP 4: GOOGLE

(**Disclaimer:** This is not how Google works, but how it used to work in the 2000s)

(Suppose it's 1990 and you type the search term "Apple" in your search engine. Before there was Google, the search engine would just find all the websites with the most number of words "Apple." But that was a problem, because there were useless websites with just the string "Apple Apple Apple Apple Apple Apple." But then Google came along and saved us all, with its amazing Pagerank algorithm)

Setting: 4 websites (1, 2, 3, 4), all with your search term in it.

(It's all based on an importance vector, which tells you which site is most important)

**Ex:** $v = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.4 \\ 0.3 \end{bmatrix}$ (Importance Vector)

Means: 3 is most important, then 4, then 1, then 2
(so in your search results, you would see 3 first, then 4, etc.)

(How to determine which one is more important? It all depends on how many sites link to your site)

Assumption: The more links there are *to* your website, the more important it is
(Analogy: The more people tweet about you, the more important you are)
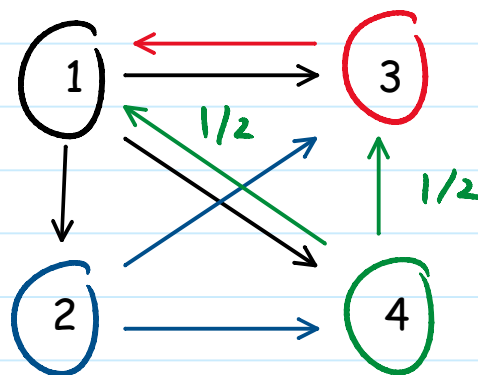
Ex:
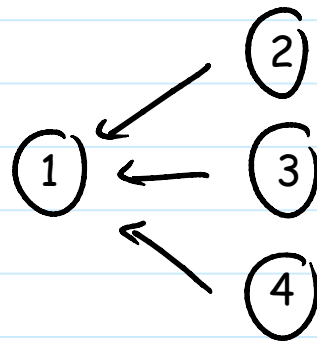
(Arrow means: "Links to," so both 2, 3, 4 have a link to 1)

Here 1 is most important

(But what about more complicated situations?)

Problem: Consider the following scenario:

Question: How to rank the sites

(Hard to figure out, especially given that you click many times! But just as before, diagonalization helps!)

Initial Vector:

$$v_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$ (= initially, all websites are equally important)

v_n = Rank after n clicks

STEP 1: Matrix

$v_{n+1} = A \; v_n$

Next   Current

$$A = \begin{array}{c c} & \begin{array}{c c c c} 1 & \quad 2 & \quad 3 & \quad 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

(Think of it in terms of probabilities)

Ex: If you start at 4, there's a 1/2 chance that you'll end up in 1 and a 1/2 chance that you'll end up in 3)

STEP 2: $A = PDP^{-1}$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Small & 0 & 0 \\ 0 & 0 & Small & 0 \\ 0 & 0 & 0 & Small \end{bmatrix} \qquad P = Nightmare$$

STEP 3:

$v_n = A^n v_0$ (n steps)

$$v_\infty = A^\infty v_0 = (P D^\infty P^{-1}) v_0 = P \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D^\infty} P^{-1} \underbrace{\begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}}_{v_0}$$

$$v_\infty = \begin{bmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{bmatrix}$$

Answer: 1 > 3 > 4 > 2

And Google is that, but with a billion websites!

II- DOT PRODUCT AND LENGTHS

Welcome to Chapter 6, which is all about dot products and lengths!

Let me first "remind" you of the dot product

Example: If $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Then $u \cdot v = (1)(3) + (2)(4) = 11$

Example:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (1)(1) + (2)(0) + (3)(1) + (4)(0) = 4$$
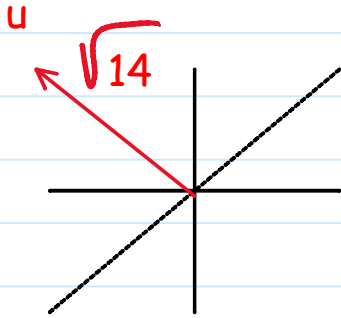
$u \qquad v$

$u^\mathsf{T} v = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (1)(1) + (2)(0) + (3)(1) + (4)(0) = 4$

So $u \cdot v = u^\mathsf{T} v$

Example: $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$||u|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

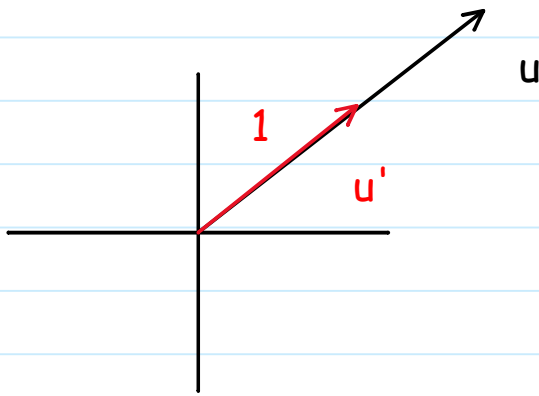$$||u|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Note: $||u|| = \sqrt{u \cdot u}$

Example: "Normalize" $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (= make it length 1)

$$||u|| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

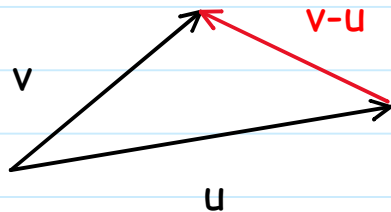Fact: $u' = \dfrac{u}{||u||}$ has length 1

Here: $u' = \dfrac{u}{||u||} = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ "normalized vector"

(Why? $\|u'\| = \|\dfrac{u}{\|u\|}\| = \|\dfrac{1}{\|u\|}\,u\| = \dfrac{1}{\|u\|}\|u\| = 1$)

Example: Distance between $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\|u - v\| = \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\| = \sqrt{4 + 4} = \sqrt{8}$$

Example:

$$\left(\frac{u \cdot v}{v \cdot v} \right) v = \left(\frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}}{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{11}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33/25 \\ 44/25 \end{bmatrix}$$

(Will be important later)

Warning: $||cu|| = |c| \, ||u||$

Ex: $||(-2)u|| = |-2| \, ||u|| = 2 \, ||u||$, NOT $(-2)||u||$
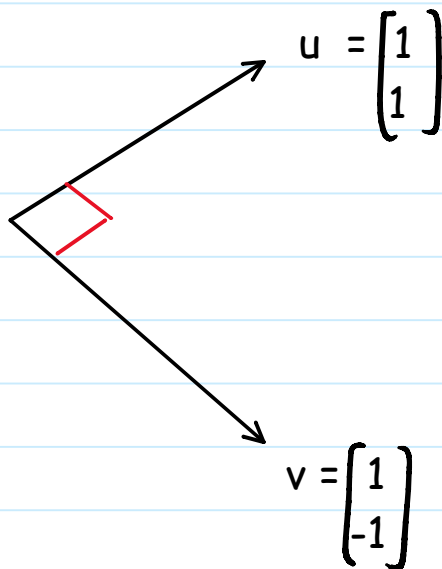
III- ORTHOGONALITY

SINGLE, MOST important concept in Chapter 6, will go VERY far with it!

Example: If $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then

$u \cdot v = (1)(1) + (1)(-1) = 0$

More importantly, notice that u and v are perpendicular!

$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Definition: u and v are orthogonal ($\perp$) If $u \cdot v = 0$

Definition: u and v are orthogonal (⊥) if u · v = 0

Ex: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ = 8 - 15 = -7, so not ⊥

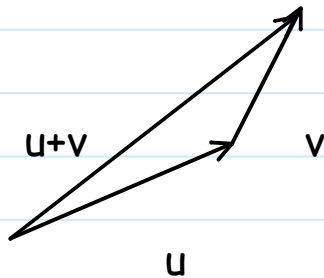**Ex:** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ = 2 - 3 + 1 = 0, so ⊥

IV- FUN (IN)EQUALITIES

Know the statement of 1 and 2 and be able to derive 2 and 3
(Also derive 1 and 4, but I would give you hints)

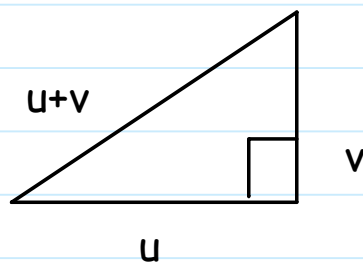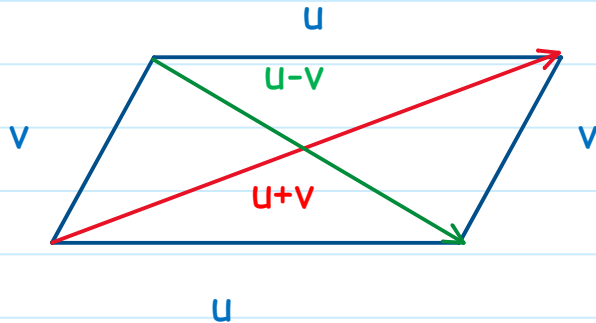### 1) TRIANGLE INEQUALITY

$$||u+v|| \leq ||u|| + ||v||$$

u+v   v

u

2) PYTHAGOREAN THEOREM

If u ⊥ v, then $||u+v||^2 = ||u||^2 + ||v||^2$

u+v

v

u

3) PARALLELOGRAM LAW

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$$

(Adding up the squares of diagonals of a parallelogram is the same as adding the squares of all the sides)

4) CAUCHY-SCHWARZ INEQUALITY

$$|u \cdot v| \leq ||u|| \, ||v||$$

Note: Compare with $u \cdot v = ||u|| \, ||v|| \cos(\theta)$ in Math 2D

Between
-1 and 1