LECTURE 23: ORTHOGONALITY-AWESOMENESS

Tuesday, November 19, 2019 5:47 PM

Today: What makes orthogonality so awesome? There will be **MANY** miracles!

I- ORTHOGONAL SETS

Definition: $\mathcal{B}$ is orthogonal if for any two <u>different</u> vectors u and v in $\mathcal{B}$, we have $u \cdot v = 0$

Example: Is $\mathcal{B} = \left\{ \underset{u}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}, \underset{v}{\begin{bmatrix} -3 \\ 2 \end{bmatrix}} \right\}$ orthogonal?

$u \cdot v = 2(-3) + 3(2) = 0$, so YES

(**Note**: No need to check $v \cdot u$ because $v \cdot u = u \cdot v = 0$
No need to check $u \cdot u$ because need <u>different</u> vectors)

Example:

(a) Is $\mathcal{B} = \left\{ \underset{u}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{v}{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}, \underset{w}{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}} \right\}$ orthogonal?

(Seems like there's a million different things to check, but really only need to check 3 things)

$$u \cdot v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$u \cdot w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 0 - 2 = 0$$

$$v \cdot w = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$$

(b) Is $\mathscr{B}$ linearly independent?

Neat Fact: Any orthogonal set without the **0** vector is linearly independent!

So YES

Why? Suppose $a\,u + b\,v + c\,w = \mathbf{0}$ (show $a = b = c = 0$)

Dot with u: $(au + bv + cw) \cdot u = \mathbf{0} \cdot U = 0$

$$a\,u \cdot u + b\,\overbrace{v \cdot u}^{0} + c\,\overbrace{w \cdot u}^{0} = 0$$

(by orthogonality)

$$a\,(\underbrace{u \cdot u}_{\neq 0}) = 0$$

$\Rightarrow a = 0$ (since $u \neq \mathbf{0}$)

Similarly, dot with v to get $b = 0$ and dot with w to get $c = 0$

Note: In particular, here $\mathscr{B}$ is a LI set in R^3 with 3 vectors, hence it's a Basis for R^3 !

Moral: Orthogonality is NICE, it simplifies a lot of things that we had trouble with before

(c) Express $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ as a linear combo of u, v, w

Find a, b, c such that $x = au + bv + cw$

(Usually a pain to do because it required row-reduction, but here none of that stuff!)

Trick: Dot with u:

$$x \cdot u = (au + bv + cw) \cdot u = a\, u \cdot u + b \underbrace{\cancel{v \cdot u}}_{0} + c \underbrace{\cancel{w \cdot u}}_{0} = a\, u \cdot u$$

$a\, u \cdot u = x \cdot u$

$$a = \frac{x \cdot u}{u \cdot u}$$

Here:

$$a = \frac{\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} = \frac{6}{2} = 3$$

Similarly: Dot with v

$$b = \frac{x \cdot v}{v \cdot v}$$

$$b = \frac{\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}} = \frac{2}{18} = \frac{1}{9}$$

$$c = \frac{x \cdot w}{w \cdot w}$$

$$c = \frac{\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}} = \frac{-4}{9}$$

Says:

$$\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + \frac{-4}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$x = a \quad u + b \quad v + c \quad w$$

(Again, imagine doing this with row-reduction, here we get a direct formula!)

Summary: If $\mathcal{B} = \{u, v, w\}$ is orthogonal and $x = au + bv + cw$, then

$$a = \frac{x \cdot u}{u \cdot u} \qquad b = \frac{x \cdot v}{v \cdot v} \qquad c = \frac{x \cdot w}{w \cdot w}$$

Make sure to know this formula, we'll use it over and over again!

Hugging analogy: Suppose x is walking down the street and just wants to hug everyone. First, x hugs u (= $x \cdot u$) and then u is so happy it hugs itself (= $u \cdot u$), then x hugs v (= $x \cdot v$) and v hugs itself (= $v \cdot v$), and finally x hugs w (= $x \cdot w$) and w hugs itself (= $w \cdot w$)

II- ORTHONORMAL SETS

Let's power up and define *orthonormal* sets (= *super* orthogonal sets), which are even nicer than orthogonal sets

Example:

(a) Is $\mathcal{B} = \left\{ \underset{u}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}, \underset{v}{\begin{bmatrix} -4 \\ 3 \end{bmatrix}} \right\}$ orthogonal?

$u \cdot v = 3(-4) + 4(3) = 0$, so yes

(b) Is $\mathcal{B}$ orthonormal?

Definition: Orthonormal = Orthogonal + Each vector has length 1

Here: $||u|| = 3^2 + 4^2 = 5$ =/= 1, so NO

(c) Normalize $\mathscr{B}$ (= make each vector length 1)

$$u' = \frac{u}{||u||} = \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}{5} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$v' = \frac{v}{||v||} = \frac{\begin{bmatrix} -4 \\ 3 \end{bmatrix}}{5} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

(d) Is $\mathscr{B}' = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \right\}$ orthonormal?

YES (Orthogonal + Length 1)

(e) Is $\mathscr{B}'$ a basis for R^2 ?

YES, since $\mathscr{B}'$ is in particular orthogonal (=> LI) and has 2 vectors

(f) Express $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combo of $\left\{ \underset{u'}{\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}, \underset{v'}{\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}} \right\}$

$x = a\,u' + b\,v'$

$$a = \frac{x \cdot u'}{u' \cdot u' = 1} = x \cdot u' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 7/5$$

$$b = \frac{x \cdot v'}{v' \cdot v'} = x \cdot v' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} = -1/5$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \left(\frac{7}{5}\right)\begin{bmatrix}3/5\\4/5\end{bmatrix} + \left(\frac{-1}{5}\right)\begin{bmatrix}-4/5\\3/5\end{bmatrix}$$ (WOW)

Summary: If $\mathscr{B} = \{u,v,w\}$ is orthonormal and $x = au + bv + cw$, then

$a = x \cdot u \qquad b = x \cdot v \qquad c = x \cdot w$ (Even simpler!)

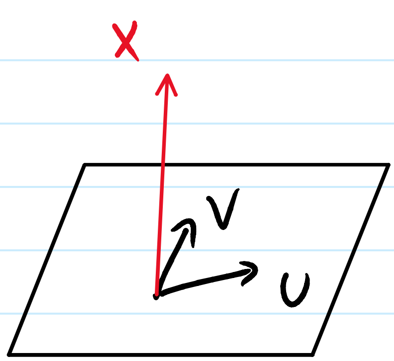
III- ORTHOGONAL COMPLEMENTS

(Completely unrelated, but *complements* our discussion of orthogonality)

Example: Find all the vectors $\mathbf{x}$ in R^3 that are orthogonal to

$$u = \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Picture:



Slow way:

Suppose $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to u and v, then:

$$\begin{cases} \mathbf{x} \cdot u = 0 \\ \mathbf{x} \cdot v = 0 \end{cases}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x + z = 0 \\ x + y = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

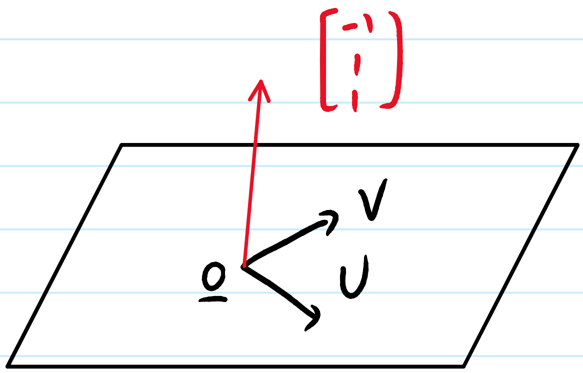
Row-reduce: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$$\begin{cases} x + z = 0 \\ y - z = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ y = z \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Answer: Span$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Picture:



Application (Math 2D):

Find the equation of the plane that goes through 0 and contains the vectors u = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and v = $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

"Normal" vector is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (see picture above)

Answer: (-1)x + (1)y + (1)z = 0 => -x + y + z = 0

(Could use cross-products, but this technique works in ANY dimension!)

Faster way:

$$A = [u\ v] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Found: $$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A^T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Ans} = \text{Nul}(A^T) = \text{Nul} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \ldots = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Note: CANNOT POSSIBLY be Nul(A), because Nul(A) is in R^2 but we need something in R^3 !