LECTURE 24: ORTHOGONAL PROJECTIONS

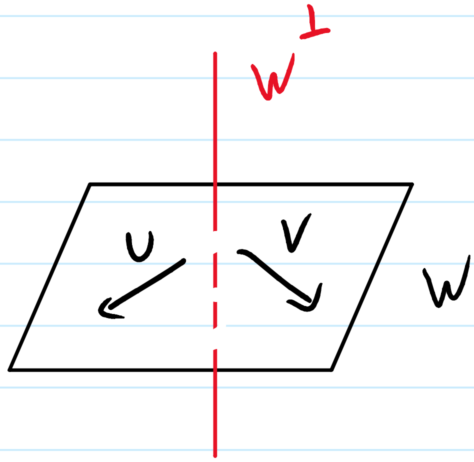
Wednesday, November 20, 2019 9:30 PM

I- $W^{\perp}$

Definition: If W is a subspace of R^n, then

$$\boxed{W^{\perp} = \text{All the vectors that are } \perp \text{ to (all the vectors) in } W}$$

Picture:



Example: If $W = \text{Span}\left\{ \begin{array}{cc} u & v \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{array} \right\}$ find (a basis for) $W^{\perp}$

Enough to find all the vectors perpendicular to u and v

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Ans

$$W^{\perp} = \text{Nul}(A^T) = \text{Nul}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \dots = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: In terms of matrices, $W = Col(A)$ and we showed $W^{\perp} = Nul(A^T)$

Hence $\boxed{(Col(A))^{\perp} = Nul(A^T)}$

Mnemonic: Put that frown upside down, put $\perp$ inside your matrix to become T

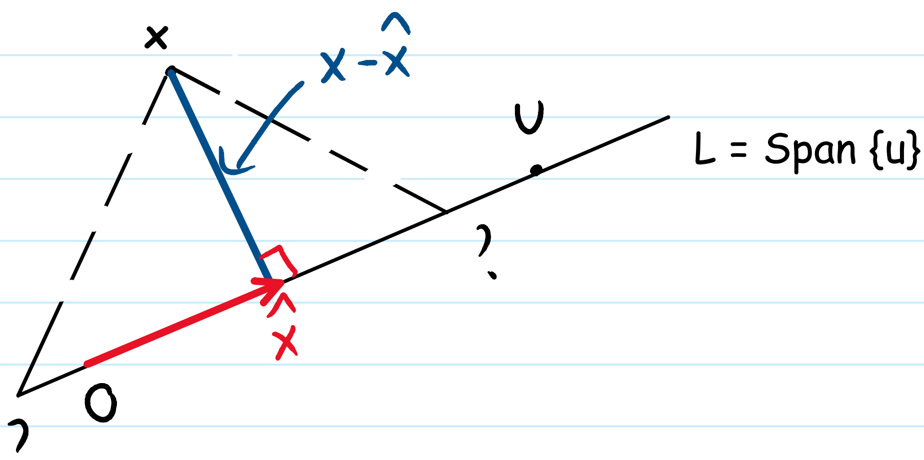
Also $\boxed{(Nul(A))^{\perp} = Col(A^T)}$ (= "Row(A)")

II- OP ON A LINE

Today: Orthogonal Projections, which is a neat way of "squishing" a vector on another vector (this will have **LOTS** of insane applications, see next 2 lectures)

Goal: Given a point **x** and a line L, want to "squish" **x** on L

Picture:



Many different ways of doing that, but there is one way that is optimal

Definition: The orthogonal projection $\hat{x}$ of x on L is

$$\underbrace{\text{the point on } L}_{(1)} \text{ such that } \underbrace{x - \hat{x} \perp L}_{(2)}$$

Example: Calculate the OP $\hat{x}$ of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ on $L = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{u} \right\}$

WTF

(1) $\hat{x}$ is in L, so $\hat{x} = a\, u$ for some a

(2) $x - \hat{x} \perp L$, so $(x - \hat{x}) \perp u$

$\Rightarrow (x - \hat{x}) \cdot u = 0$

$\Rightarrow (x - au) \cdot u = 0$

$\Rightarrow x \cdot u - a\, u \cdot u = 0$

$\Rightarrow a\, u \cdot u = x \cdot u$

$\Rightarrow a = \dfrac{x \cdot u}{u \cdot u}$ (OMG, the hugging formula!!!)

FACT: $\hat{x} = \underbrace{\left(\dfrac{x \cdot u}{u \cdot u} \right)}_{a} u$

Here: $\hat{x} = \left(\dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

Do NOT blindly memorize this! **ALWAYS** think: $\hat{x}$ is some multiple of u ($x = au$), where the multiple is given by hugging!

(b) Calculate the (smallest) distance between x and L

(**Don't** memorize, look at the picture above !)

<u>Ans</u>: $||x - \hat{x}|| = \left|\left| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix} \right|\right| = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}}$

(So in this sense, $\hat{x}$ is optimal -> 6.5)

(c) Find a vector perpendicular to $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

<u>Ans</u>: $x - \hat{x} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}$

(Gives an easy way of building perpendicular vectors -> 6.4)

(d) Write x as a sum of 2 vectors, one in L and one perpendicular to L

Trick: $x = \hat{x} + (x - \hat{x}) = \underbrace{\begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}}_{\text{in } L} + \underbrace{\begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}}_{\perp \text{ to } L}$

(Similar in physics to decomposing a force into tangential and normal component)

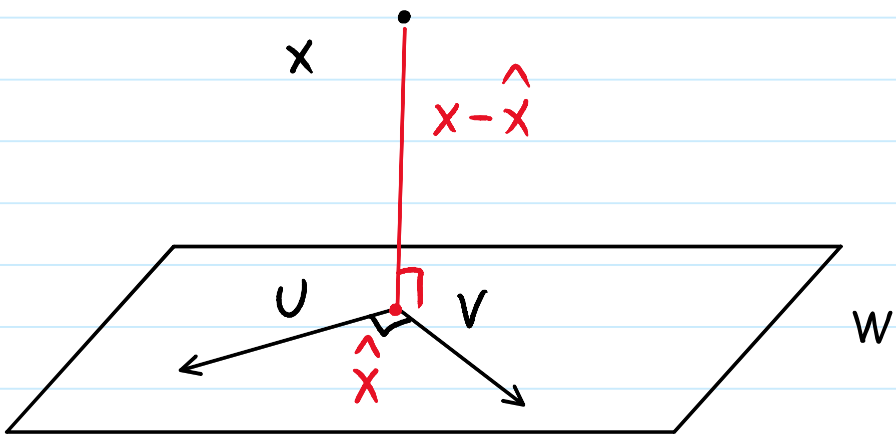
III- OP ON ≥ 2 VECTORS

The beautiful thing is that we can do the same procedure not only for lines, but also for planes (and really any subspace)

Example: Let $W = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{u}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v} \right\}$ $\qquad x = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$

MUST BE $\perp$!!!

(a) Find the OP $\hat{x}$ of x on W



Definition: $\underbrace{\hat{x} \text{ is the vector in } W}_{(1)}$ such that $\underbrace{(x - \hat{x}) \perp u \,\&\, v}_{(2)}$

By (1), $\hat{x} = au + bv$

By (2), $\hat{x} = \left(\dfrac{x \cdot u}{u \cdot u}\right)u + \left(\dfrac{x \cdot v}{v \cdot v}\right)v$

$$= \left(\frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}} \right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= (3/2)\begin{bmatrix}1\\1\\0\end{bmatrix} + (5/2)\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1\\4\\0\end{bmatrix}$$

(b) Find the distance between x and W

$$||x - \hat{x}|| = \left|\left| \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}-1\\4\\0\end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix}0\\0\\3\end{bmatrix} \right|\right| = 3$$

(c) Find a vector $\perp$ to u and v

$$x - \hat{x} = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

(d) Write x as the sum of 2 vectors, one in W and one $\perp$ to W

$$x = \underbrace{\hat{x}}_{\text{in } W} + \underbrace{(x - \hat{x})}_{\text{in } W^{\perp}}$$

IV- ORTHOGONAL MATRICES

Related to this is the concept of orthogonal matrices

Example: Let $Q = \begin{array}{c} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \\ \begin{array}{cc} u \quad\quad & \quad\quad v \end{array} \end{array} = [u \mid v]$

NOTICE: The columns of Q are orthonormal

(a) Find $Q^T Q$ $= I$ Orthonormal

Why? $Q^T Q = \begin{bmatrix} u \\ \hline v \end{bmatrix} [u \mid v] = \begin{bmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(b) Find $QQ^T x$ where $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

WARNING: In general, $QQ^T \neq I$

$$
\begin{aligned}
Q Q^T x &= [u \mid v] \begin{bmatrix} \boxed{u} \\ \hline v \end{bmatrix} \begin{bmatrix} \boxed{x} \end{bmatrix} \\
&= \begin{bmatrix} \boxed{u} & v \end{bmatrix} \begin{bmatrix} u \cdot x \\ v \cdot x \end{bmatrix} \\
&= (u \cdot x) \boxed{u} + (v \cdot x) v \\
&= \left(\frac{x \cdot u}{\underbrace{u \cdot u}_{=1}} \right) u + \left(\frac{x \cdot v}{\underbrace{v \cdot v}_{=1}} \right) v \\
&= \hat{x} \text{ !!!}
\end{aligned}
$$

$\boxed{\text{Fact: } QQ^T x = \hat{x} = \text{OP on } W = \text{Span}\{u,v\} = \text{Col}(Q)}$

Here:

$$QQ^T x = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \ldots = \begin{bmatrix} 1 \\ 2/3 \\ 1/6 \end{bmatrix}$$

(c) Calculate $||Qx||$

FACT: $||Qx|| = ||x||$ $\quad = \left|\left| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right|\right| = \sqrt{2}$

Why? $||Qx||^2 = (Qx) \cdot (Qx) = (Qx)^T Qx = x^T Q^T Q x = x^T I x$
$= x^T x = x \cdot x = ||x||^2$