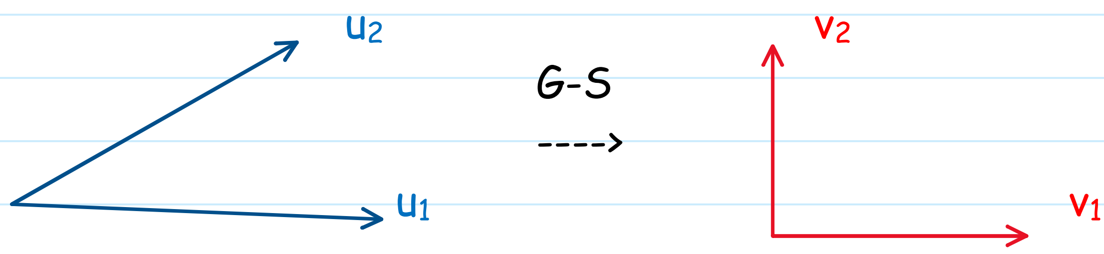
LECTURE 25: THE GRAM-SCHMIDT PROCESS

Friday, November 22, 2019 3:28 PM

Welcome to our first application of Orthogonal Projections!

Goal: Turn any (LI) set into an orthogonal set (with the same span)

Picture:



Good news! In theory, you already know how to do this :)

Note: Please don't memorize the formula, just do it geometrically!

I- GRAM-SCHMIDT WITH 2 VECTORS

Example: Find an orthogonal basis for

$$W = \text{Span}\left\{ \underbrace{\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}}_{u_2} \right\}$$

Answer: $\{v_1 , v_2\}$ ---> How to find v_1, v_2 ?

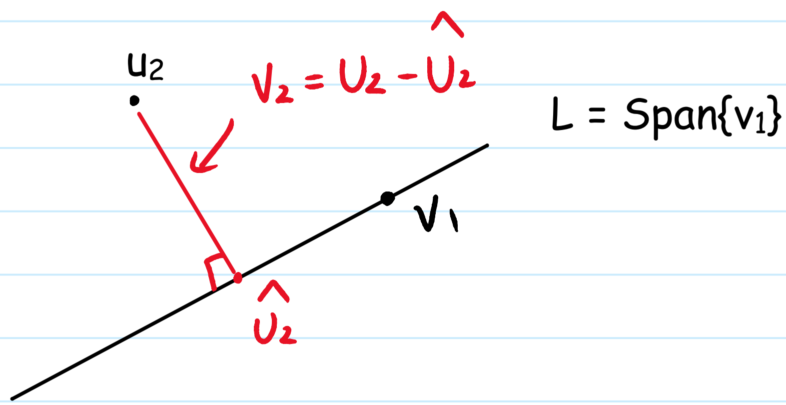
STEP 1: Gotta start with something!

$$v_1 = u_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

IMPORTANT: Once you used u_1, cross it out from your list, won't need it any more!

STEP 2: Find a vector perpendicular to v_1

Picture:



(a) Calculate $\hat{u}_2$ (= OP of u_2 on Span$\{v_1\}$)

$$\hat{u}_2 = \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= \frac{\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$= \left(\frac{10}{20}\right)\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

WARNING: DO **NOT** rescale this!

(b) $v_2 = u_2 - \hat{u}_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$ <--- THIS is ok to rescale

Ans: $\left\{ \underset{v_1}{\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}} \right\}$ (orthogonal basis for W)

Check: $v_1 \cdot v_2 = 0$ ✓

II- GRAM-SCHMIDT WITH ≥ 3 VECTORS

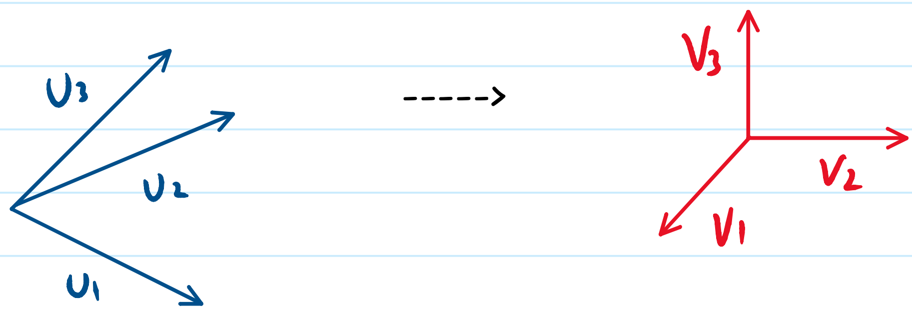
Once you know how to do it with 2 vectors, you can also do it with **any** number of vectors

Example:

(a) Find an orthogonal basis for

$$W = \text{Span}\left\{ \begin{array}{ccccc} \cancel{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} & , & \cancel{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}} & , & \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ u_1 & & u_2 & & u_3 \end{array} \right\}$$

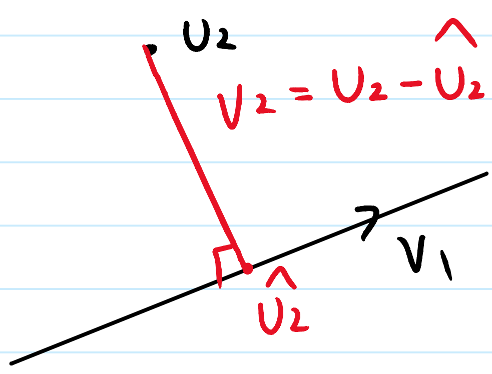
Picture:



STEP 1:

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ (Cross out u_1)

STEP 2:



$$\hat{u}_2 = \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

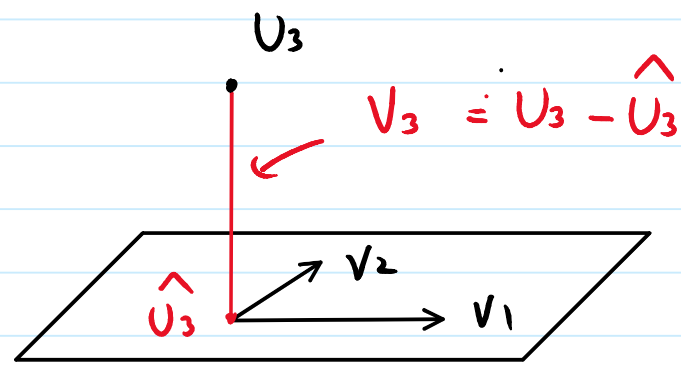
$$= (1/2) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$ (Do NOT rescale)

$$v_2 = u_2 - \hat{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$ (Rescale)

Important: Check $v_2 \cdot v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$

And cross out u_2, will never use it again

STEP 3



$$\hat{u}_2 = \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \frac{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\0\end{bmatrix}} \begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\2\\1\\2\end{bmatrix}}{\begin{bmatrix}-1\\2\\1\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\2\\1\\2\end{bmatrix}} \begin{bmatrix}-1\\2\\1\\2\end{bmatrix}$$

$$= (-1/2) \begin{bmatrix}1\\0\\1\\0\end{bmatrix} + (3/10) \begin{bmatrix}-1\\2\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}-4/5\\3/5\\-1/5\\3/5\end{bmatrix}$$

$$v_3 = u_3 - \hat{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 3/5 \\ -1/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -3/5 \\ 1/5 \\ 2/5 \end{bmatrix} \sim \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix}$$

Check

$$v_3 \cdot v_1 = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$v_3 \cdot v_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 1 - 6 + 1 + 4 = 0$$

Answer: $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix} \right\}$

(b) Find an ortho**normal** basis for W

Answer: $\{w_1, w_2, w_3\}$

$$w_1 = \frac{v_1}{||v_1||} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} -1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$$

$$w_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} -1/\sqrt{15} \\ -3/\sqrt{15} \\ 1/\sqrt{15} \\ 2/\sqrt{15} \end{bmatrix}$$

POINT: From now on it's easy to create orthonormal sets!

III- QR DECOMPOSITION

Application of G-S to matrices, will see next time why it's useful

Example: Find the QR-decomposition of

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Idea: Write $A = \underbrace{Q}_{\substack{\text{Orthonormal} \\ \text{Columns} \\ \text{(like last time)}}} \underbrace{R}_{\text{Upper triangular}}$

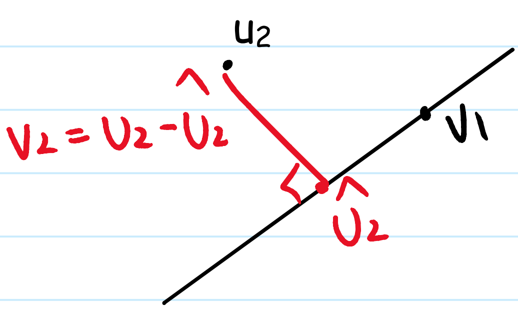
TRICK: Apply G-S to $\left\{ \underset{u_1}{\begin{bmatrix}2\\2\\1\end{bmatrix}}, \underset{u_2}{\begin{bmatrix}3\\4\\1\end{bmatrix}} \right\}$ (Columns of A)

STEP 1:

$$v_1 = u_1 = \begin{bmatrix}2\\2\\1\end{bmatrix}$$

STEP 2:

Picture:



$$\hat{u}_2 = \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= \frac{\begin{bmatrix}3\\4\\1\end{bmatrix} \cdot \begin{bmatrix}2\\2\\1\end{bmatrix}}{\begin{bmatrix}2\\2\\1\end{bmatrix} \cdot \begin{bmatrix}2\\2\\1\end{bmatrix}} \begin{bmatrix}2\\2\\1\end{bmatrix}$$

$$= \left(\frac{15}{9}\right)\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10/3 \\ 10/3 \\ 5/3 \end{bmatrix}$$

$$v_2 = u_2 - \hat{u}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 10/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\{v_1, v_2\} = \left\{ \underset{v_1}{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}} \right\} \quad (\text{Check } v_1 \cdot v_2 = 0)$$

STEP 3: $\{w_1, w_2\}$ orthonormal

$$w_1 = \frac{v_1}{||v_1||} = \frac{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}{\sqrt{9}} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$w_2 = \frac{v_2}{||v_2||} = \frac{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{9}} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Answer:

$$Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}$$ (orthonormal columns)

Find R:

Trick: $A = QR$

$\Rightarrow Q^T A = Q^T Q R = I R = R$

$$\Rightarrow R = Q^T A = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}$$

(Next time we'll see why it's so useful)