LECTURE 27: FINAL EXAM REVIEW (I)

Monday, November 25, 2019 3:36 PM

Congratulations, we are done with the course, so the next 3 lectures will be review. There is just one little thing I have to wrap up about orthogonal matrices (which will be relevant today)

I- ORTHOGONAL MATRICES

Definition: If Q is **SQUARE** and has orthoNORMAL columns, then Q is an orthoGONAL matrix

Example: $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(a) Find Q^{-1}

$Q^T Q = I$ AND Q is SQUARE

$\Rightarrow Q^{-1} = Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(b) Find QQ^T

Since $Q^{-1} = Q^T \Rightarrow QQ^T = QQ^{-1} = I \Rightarrow QQ^T = I$

Important: Q has to be square! In general, $QQ^T \neq I$, $QQ^T x$ is OP of x on Col(Q))

(c) Find det(Q)

$Q^T Q = I \Rightarrow \det(Q^T Q) = \det(I)$

$\Rightarrow \det(Q^T)\det(Q) = 1$
$\Rightarrow \det(Q)\det(Q) = 1$
$\Rightarrow (\det(Q))^2 = 1$
$\Rightarrow \det(Q) = \pm 1$

(d) Calculate $||Qx||$, $\quad x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

FACT: $||Qx|| = ||x||$

Why?

$$\begin{aligned} ||Qx||^2 &= (Qx) \cdot (Qx) && (||u||^2 = u \cdot u) \\ &= (Qx)^T (Qx) && (u \cdot v = u^T v) \\ &= x^T Q^T Q x && ((AB)^T = B^T A^T) \\ &= x^T I x \\ &= x^T x \\ &= ||x||^2 \end{aligned}$$

$$\Rightarrow ||Qx|| = ||x|| = \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{25} = 5$$

Note: This last fact is also true for nonsquare matrices.

For the rest of today, we'll review eigenvectors and orthogonality (and I'll sneak in something useful :))

II- EIGENVALUES

Example: Find the eigenvalues of

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Aside: Notice that A is symmetric ($A^T = A$), so today's theme is about symmetric matrices (and what makes them special)

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & 2 & -4 \\ 2 & \lambda-6 & -2 \\ -4 & -2 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-3)\begin{vmatrix} \lambda-6 & -2 \\ -2 & \lambda-3 \end{vmatrix} -2 \begin{vmatrix} 2 & -2 \\ -4 & \lambda-3 \end{vmatrix} -4 \begin{vmatrix} 2 & \lambda-6 \\ -4 & -2 \end{vmatrix}$$

$$= (\lambda-3)[(\lambda-6)(\lambda-3)-4] -2[2(\lambda-3)-8] -4[-4+4(\lambda-6)]$$
$$= (\lambda-3)[\lambda^2 - 9\lambda + 18 - 4] -2[2\lambda - 6 - 8] -4[-4 + 4\lambda - 24]$$
$$= (\lambda-3)[\lambda^2 - 9\lambda + 14] -2[2\lambda - 14] -4[4\lambda - 28]$$
$$= (\lambda-3)(\lambda-7)(\lambda-2) -4(\lambda-7) - 16(\lambda-7)$$

$$= (\lambda-7)[(\lambda-3)(\lambda-2)-4-16]$$
$$= (\lambda-7)[\lambda^2 - 5\lambda + 6 - 20]$$
$$= (\lambda-7)(\lambda^2 - 5\lambda - 14)$$
$$= (\lambda-7)(\lambda-7)(\lambda+2)$$
$$= (\lambda-7)^2(\lambda+2) = 0$$

$\Rightarrow \boxed{\lambda = 7 \text{ and } \lambda = -2}$

III- DIAGONALIZATION

Example: With A as above, find D and P with $A = PDP^{-1}$

$$\boxed{D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}}$$

$\underline{\lambda = 7}$

$$\mathrm{Nul}(7I-A) = \mathrm{Nul}\begin{bmatrix} 7-3 & 2 & -4 \\ 2 & 7-6 & -2 \\ -4 & -2 & 7-3 \end{bmatrix}$$

$$= \mathrm{Nul}\begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}$$

$$= \mathrm{Nul}\begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \mathrm{Nul}\begin{array}{c} \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} & \uparrow & \uparrow \\ & y & z \end{array} \end{array}$$

$x + (1/2)\, y - z = 0 \Rightarrow x = (-1/2)y + z$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2\, y + z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$E_7 = \mathrm{Span}\left\{ \underset{(\times 2)}{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(Please rescale!)

(Please rescale!)

Similarly

$$E_{-2} = \text{Span}\left\{\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}\right\}$$

$$\boxed{P = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}$$

$$\quad (7) \quad (7) \quad (-2)$$

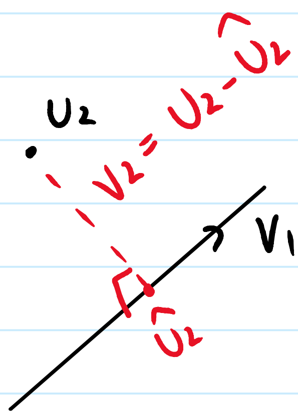
IV- GRAM-SCHMIDT

(What happens if we apply Gram-Schmidt to each eigenspace?)

Example:

(a) Find an orthonormal basis for $E_7 = \text{Span}\left\{\underset{u_1}{\cancel{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}}, \underset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}\right\}$

$$v_1 = u_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$ (cross out u_1)



$$\hat{u}_2 = \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = \frac{\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\2\\0\end{bmatrix}}{\begin{bmatrix}-1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\2\\0\end{bmatrix}} \begin{bmatrix}-1\\2\\0\end{bmatrix} = \frac{-1}{5} \begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}1/5\\-2/5\\0\end{bmatrix}$$

(do not rescale)

$$v_2 = u_2 - \hat{u}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/5\\-2/5\\0\end{bmatrix} = \begin{bmatrix}4/5\\2/5\\1\end{bmatrix} \sim \begin{bmatrix}4\\2\\5\end{bmatrix}$$ (ok to rescale)

Check: $v_1 . v_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}4\\2\\5\end{bmatrix} = 0$

$$w_1 = \frac{v_1}{||v_1||} = 1/\sqrt{5} \begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}$$

$$w_2 = \frac{v_2}{||v_2||} = 1/\sqrt{45} \begin{bmatrix}4\\2\\5\end{bmatrix} = \begin{bmatrix}4/\sqrt{45}\\2/\sqrt{45}\\5/\sqrt{45}\end{bmatrix}$$

(b) Find an orthonormal basis for E_{-2} = Span $\left\{ \underbrace{\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}}_{u_1} \right\}$

$$v_1 = u_1 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$ (Note: $\{v_1\}$ is orthogonal!)

$$w_1 = \frac{v_1}{||v_1||} = (1/3) \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

V- SYMMETRIC MATRICES

Example: What can you say about

$$P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix} ?$$

$$\begin{array}{ccc} (7) & (7) & (-2) \end{array}$$

Columns of P are orthonormal

=> P is an orthogonal matrix

=> $P^{-1} = P^T$

Aside:

1) This is really cool! After doing G-S, we know that for *each* eigenspace, the vectors are orthonormal, but there's no reason why all 3 of them have to be orthonormal, but here for symmetric matrices it's true!

2) Also, usually for matrices, you only have $A = PDP^{-1}$, but with this technique (for symmetric matrices), you get $A = PDP^T$ (this is called *orthogonally* diagonalizable)

Example: Show $B = PDP^T$ is always symmetric ($B^T = B$)

$$B^T = (PDP^T)^T = (P^T)^T D^T P^T = P D^T P^T = PDP^T = B$$

($D^T = D$ since D is diagonal)

(So symmetric is the SAME as orthogonally diagonalizable, WOW)