Math 112A — Homework 5

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Reading: Sections 3.1 and 3.2. I will probably have started with 4.1 already, but you won’t be responsible for it until Homework 6. In section 3.2, do NOT read the section on ‘The Finite Interval,’ we won’t cover that. Also, do NOT memorize all the formulas for solutions like (6) in 3.1, but definitely be able to derive them!

Note: In some of the problems, by ‘Solve,’ the book means write the answer in terms of the error function

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-y^2} dy \]

- **Section 3.1**: 1 (see hint), 2 (see hint), 3, AP1, AP2, AP3
- **Section 3.2**: 1 (what they mean is \( u_x(0, t) = 0 \)), 5 (ignore the part about singularity), AP4

Additional Problems:

AP 1: Use your answer in Problem 3 in 3.1 to derive the solution formula for the following half-line Neumann problem for \( x > 0 \). Do NOT simplify the resulting expression, but please write it in terms of \( \phi \).

\[
\begin{align*}
  w_t &= kw_{xx} \\
  w_x(0, t) &= 3 \\
  w(x, 0) &= \phi(x)
\end{align*}
\]
**Hint:** Let \( v(x, t) = w(x, t) - ? \) where this time \(?\) is a function whose \( x\)–derivative is 3.

**AP 2:** The following problem has nothing to do with PDEs, but illustrates something fun about even and odd functions

(a) Show that any function \( f(x) \) can be written as a sum of an even function and an odd function\(^1\)

**Hint:**

\[
f(x) = \left( \frac{f(x) + f(-x)}{2} \right) + \left( \frac{f(x) - f(-x)}{2} \right)
\]

(b) What do you get in the case \( f(x) = e^x \)? Do those functions have a special name? We’ll see them again in Chapter 6.

**AP 3:** Let \( f \) be any function defined on \((0, \pi)\).

(a) Express the following quantity (called the Fourier cosine coefficient, see Chapter 5) in terms of \( \int_{0}^{\pi} f(x) \cos(nx)dx \)

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f_{\text{even}}(x) \cos(nx)dx
\]

Here \( f_{\text{even}} \) is the even extension of \( f \) on the interval \((-\pi, \pi)\)

(b) Same as (a), but replace \( \cos \) with \( \sin \) and \( f_{\text{even}} \) with \( f_{\text{odd}} \).

**AP 4:**

(a) Solve the following wave equation on the half-line \( x > 0 \). Do NOT simplify the resulting expression, but please write everything in terms of \( \phi \) and \( \psi \).

\[
\begin{align*}
w_{tt} &= c^2 w_{xx} \\
w(0, t) &= 1 \\
w(x, 0) &= \phi(x) \\
w_t(x, 0) &= \psi(x)
\end{align*}
\]

\^1In fact this decomposition is unique, but you don’t have to show that
(b) Use the result of Problem 1 in 3.2 to solve the following Neumann problem on the half-line $x > 0$. Do NOT simplify the resulting expression, but please write everything in terms of $\phi$ and $\psi$.

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2} \\
\frac{\partial w}{\partial x}(0, t) &= 3 \\
w(x, 0) &= \phi(x) \\
\frac{\partial w}{\partial t}(x, 0) &= \psi(x)
\end{align*}
\]

**Hint for 1 in 3.1:** I know, this problem is a bit long 😊

My suggestion is, instead of starting with the definition of $S$, first write everything in terms of $S(x - y)$ and $S(x + y)$ then use the u-sub $p = x - y$ (for the first integral), and $q = x + y$ (for the second one), and then only use the explicit formula of $S$ and do the completing the square trick. Also remember that by solving, the book means write the answer in terms of $\text{Erf}(x)$.

**Hint for 2 in 3.1:** There isn’t much you need to do, because you can use the result of Example 1 (with suitable modifications)