MIDTERM SOLUTIONS

Sunday, October 27, 2019 1:31 PM

PROBLEM 1: (a) True, (b) False, (c) True, (d) False, (e) False

OPTIONAL Justifications:

(a) A has at most 4 pivots, hence at least 3 free variables

(b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Then there is 1 free variable (=> $Ax = 0$ has a nonzero solution)
BUT also a pivot in every row (=> $Ax = b$ is consistent for all b)

(c) T onto => A (= matrix of T) is invertible (by the IMT)
=> T is one-to-one (by the IMT again)

(d) $\det(3I) = \det\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 27$ but $3\det(I) = 3\,(1) = 3$

(e) $A\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\underline{\underline{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}}$

PROBLEM 2:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -5 & -1 & 7 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \quad \times 5 \quad \times -3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -5 & -1 & 7 & 1 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \quad \times 1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ 0 & -1 & -3 & 1 & 0 & 5 \end{array}\right]$$

$$\longrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \quad \times -4 \quad \times 2$$

$$\longrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 5 \\ 0 & 1 & 0 & -4 & -3 & -11 \\ 0 & 0 & 1 & 1 & 1 & 8 \end{array}\right]$$

$$\underbrace{\hphantom{1\;0\;0}}_{I} \qquad \underbrace{\hphantom{2\;2\;5}}_{A^{-1}}$$

Hence $\boxed{A^{-1} = \begin{bmatrix} 2 & 2 & 5 \\ -4 & -3 & -11 \\ 1 & 1 & 8 \end{bmatrix}}$

PROBLEM 3:

$$A = \begin{bmatrix} 2 & -6 & 12 & 4 & 40 \\ -2 & 6 & -11 & -8 & -45 \\ 4 & -12 & 24 & 10 & 84 \\ -2 & 6 & -12 & -4 & -40 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 6 & 2 & 20 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (\div 2)$$

(a) Continue row-reducing:

$$\to \begin{bmatrix} 1 & -3 & 6 & 2 & 20 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\times 4) \quad (\times -2)$$

$$\to \begin{bmatrix} 1 & -3 & 6 & 0 & 16 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\times -6)$$

$$\to \begin{array}{c} \begin{bmatrix} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & \uparrow & & & \uparrow \\ & y & & & s \end{array} \end{array}$$

$$\Longrightarrow \begin{cases} x - 3y - 2s = 0 \\ z + 3s = 0 \\ t + 2s = 0 \end{cases} \Longrightarrow \begin{cases} x = 3y + 2s \\ z = -3s \\ t = -2s \end{cases}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} 3y + 2s \\ y \\ -3s \\ -2s \\ s \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Basis for Nul}(A): \quad \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

(b) Dim(Nul(A)) = 2

(c) Pivots in Columns 1, 3, 4, hence:

$$\boxed{\text{Basis for Col}(A): \quad \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 12 \\ -11 \\ 24 \\ -12 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ 10 \\ -4 \end{bmatrix} \right\}}$$

(d) Rank(A) = 3 (= # of Pivots)

(e) dim(Nul(A)) + rank(A) = n

PROBLEM 4:

(a) There are at most 3 pivots, so not a pivot in every column, hence never one to one (by Column Theorem)

Answer: $\boxed{\text{No h}}$

(b) Row-reduce A

$$\begin{array}{c} (x-2) \\ (x-3) \end{array} \begin{bmatrix} 1 & -3 & -2 & 2 \\ 2 & -6 & -3 & 7 \\ 3 & -9 & -6 & h \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & h-6 \end{bmatrix}$$

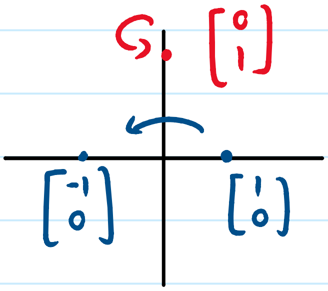
Need: Pivot in every row => h - 6 ≠ 0 => h ≠ 6

PROBLEM 5:

(a) TRUE

$$\begin{aligned} & Q^T Q = I \\ \Rightarrow\ & \det(Q^TQ) = \det(I) \\ \Rightarrow\ & \det(Q^T)\det(Q) = 1 \\ \Rightarrow\ & \det(Q)\det(Q) = 1 \\ \Rightarrow\ & (\det(Q))^2 = 1 \\ \Rightarrow\ & \det(Q) = \pm 1 \end{aligned}$$

(b) FALSE



$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

PROBLEM 6:

(a) $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ $\quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$ $\quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(b) $$\begin{aligned} \text{Area}(R') &= |\det(A)|\ \text{Area}(R) \\ &= \left|\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right| \pi 1^2 \\ &= |ab|\ \pi \\ &= \pi ab \quad (\text{since } a, b > 0) \end{aligned}$$