LECTURE 27: THE LAPLAST ONE

Monday, November 25, 2019 5:23 PM

Today: Three little remaining topics related to Laplace

I- 3 DIMENSIONS

Previously: Solved Laplace's equation in 2 dimensions by converting it into polar coordinates. The same idea works in 3 dimensions if you use spherical coordinates.

Suppose $u = u(x,y,z)$ solves $u_{xx} + u_{yy} + u_{zz} = 0$ in R^3

Using spherical coordinates, you eventually get

$$u_{rr} + \frac{2}{r} u_r + \text{JUNK} = 0$$

(where $r = \sqrt{x^2 + y^2 + z^2}$ and JUNK doesn't depend on r)

If we're looking for radial solutions, we set JUNK = 0

Get $u_{rr} + 2 \frac{u_r}{r} = 0$ which you can solve to get

$$\boxed{u(x,y,z) = \frac{-C}{r} + C' \text{ solves } u_{xx} + u_{yy} + u_{zz} = 0}$$

Finally, setting $C = -1/(4\pi)$ and $C' = 0$, you get

$$\boxed{\text{Fundamental solution of Laplace for } n = 3 \qquad S(x,y,z) = \frac{1}{4\pi r} = \frac{1}{4\pi\sqrt{x^2+y^2+z^2}}}$$

Note: In n dimensions, get $u_{rr} + \frac{n-1}{r} u_r = 0$

$$\Rightarrow \quad u(r) = \frac{-C}{r^{n-2}} + C'$$

$$\Rightarrow \quad \boxed{S(x) = \frac{Blah}{r^{n-1}}} \qquad \text{(for some complicated Blah)}$$

Why fundamental? Because can build up other solutions from this!

Fun Fact: A solution of $-\Delta u = f$ (Poisson's equation) in R^n is

$$u(x) = S(x) * f(x) = \int_{\mathbb{R}^N} S(x-y) f(y) \, dy$$

(Basically the constant is chosen such that $-\Delta S = \delta_0$ <- Dirac at 0)

II- DERIVATION OF LAPLACE

Two goals: Derive Laplace's equation, and also highlight an important structure of $\Delta u = 0$

A) SETTING

Definition: If $F = (F_1, \ldots, F_n)$ is a vector field in R^n, then

$$\boxed{\operatorname{div}(F) = (F_1)_{x1} + \ldots + (F_n)_{xn}}$$

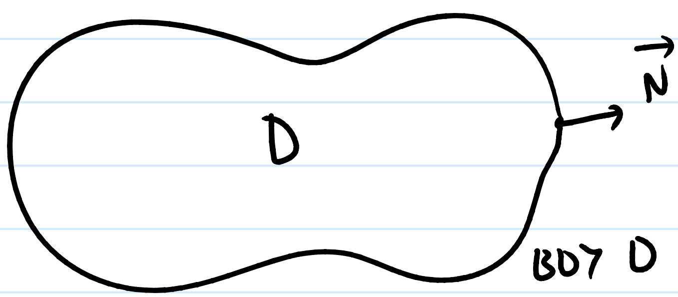
Notice: If $u = u(x_1, \ldots, x_n)$, then $\nabla u = (u_{x1}, \ldots, u_{xn})$

$\Rightarrow \operatorname{div}(\nabla u) = (u_{x1})_{x1} + \ldots + (u_{xn})_{xn} = u_{x1\,x1} + \ldots + u_{xn\,xn} = \Delta u$

Fact: $\boxed{\Delta u = \operatorname{div}(\nabla u)}$ "divergence structure"

In particular, Laplace's equation works very well with the divergence theorem

Divergence Theorem: $$\boxed{\int_{\text{bdy } D} F \cdot n \, dS = \int_D \operatorname{div}(F)\, dx}$$



B) DERIVATION

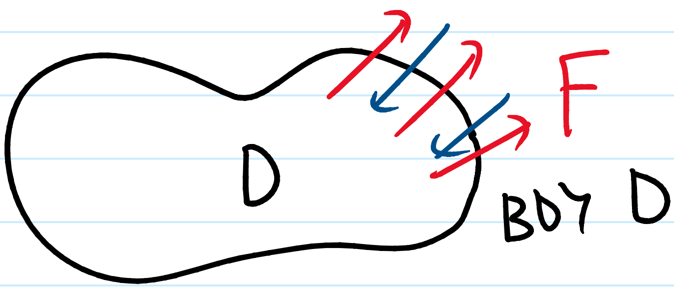
Suppose you have a fluid F that is in equilibrium
(think F = temperature or chemical concentration)

Equilibrium means that for any region D, the net flux of F is 0.

That is, for any region D of R^n

$$\int_{\text{Bdy } D} F \cdot n \, dS = 0$$

Picture: (Think Traffic In = Traffic Out, so on average it's 0)

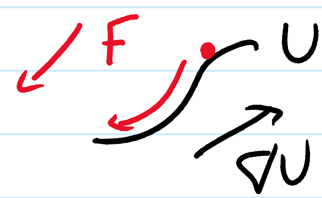


By the divergence theorem, get

$$\int_{\text{BDY } D} F \cdot n \, dS = \int_{D} \operatorname{div}(F)\, dx = 0$$

This holds for any region D, therefore $\boxed{\operatorname{div}(F) = 0}$

Now in applications, we have: $\boxed{F = -c \nabla u}$
for some density u and some $c > 0$



(Interpretation: F is proportional to the rate of change of u, but points in the opposite direction since the flow is from regions of higher to lower concentration)

(Note: Depending on the context, this is called Fick's law of diffusion, Fourier's law of heat conduction, or Ohm's law of electrical conduction)

Therefore div(F) = 0

$$\Rightarrow \operatorname{div}(-c\,\nabla u) = 0$$

$$\Rightarrow \cancel{-c}\,\operatorname{div}(\nabla u) = 0$$

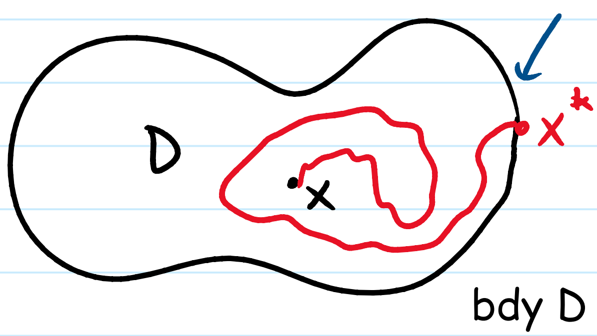
$$\Rightarrow \boxed{\Delta u = 0}$$

Note: The **SAME** proof can be adapted to derive the heat equation and even the wave equation!

III- OMG APPLICATION

Saved the best for last :)

Gain/Loss = $g(x^*)$



Suppose you start at x in D and you perform Brownian (= drunken) motion until you hit the bdy D, at which point you have a gain/loss $g(x^*)$ (think hitting a wall, and $g(x^*)$ = price you have to pay for damages)

In general, this is a random variable, so

Let $\boxed{u(x) = \text{Expected (=Average) Gain/Loss starting at } x}$

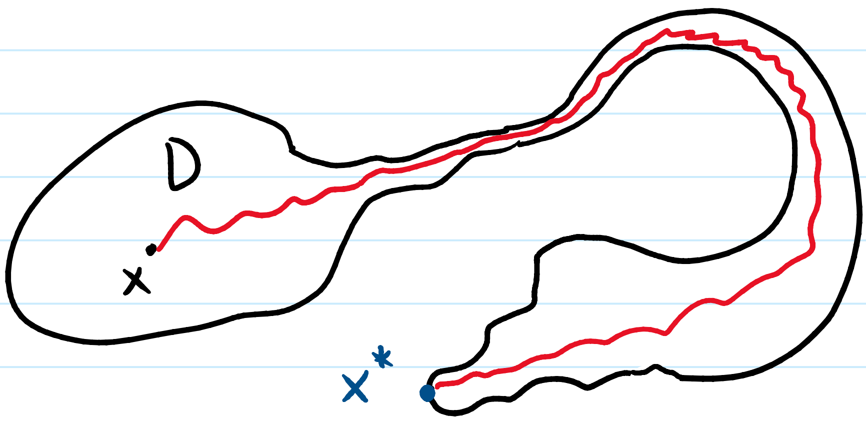
FACT: u solves $\begin{cases} \Delta u = 0 \text{ in } D \\ u = g \text{ on bdy } D \end{cases}$

Recall: Positivity: If u solves the above and $g \geq 0$ with $g \not\equiv 0$
Then $u > 0$ everywhere

INSANE CONSEQUENCE:

Suppose g is zero everywhere, but $g(x^*) > 0$ for some point x^* that is far, far away (think x^* = treasure/jackpot)

Picture:

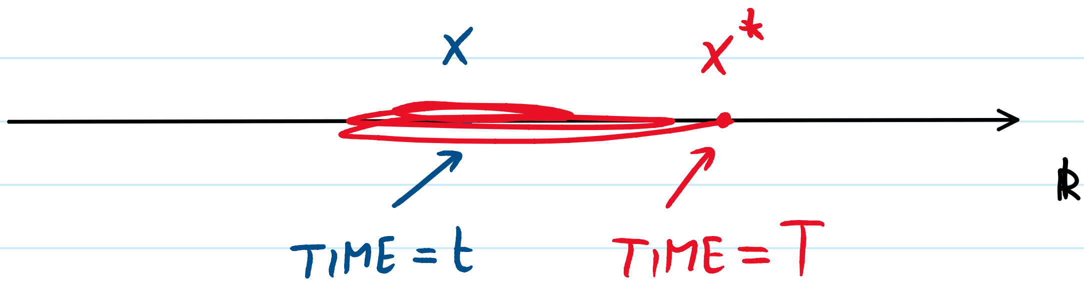


Then by positivity $u > 0$ EVERYWHERE

In particular, for all x, no matter how far, there is always a positive chance of hitting x^*, that is of finding the treasure!

The insane thing is not that there is *some* way of finding the treasure, but that there is a *positive probability* of finding it (so actually **MANY** ways of finding it)

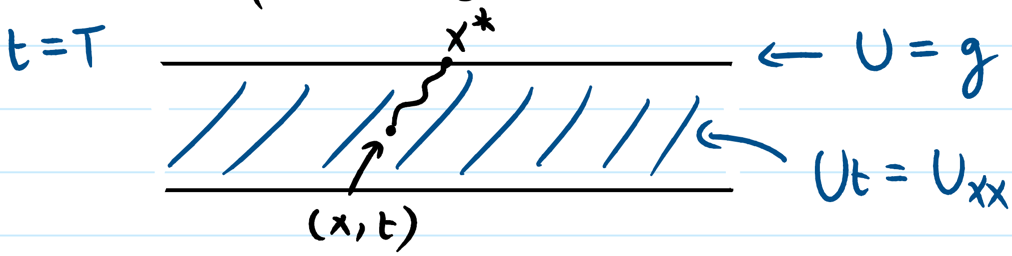
Note: There is a similar interpretation with the heat equation



Suppose you start at position x and time t (here D = ℝ), you perform Brownian motion, and at some **FIXED** time T, I tell you "STOP," and I give you a gain/loss $g(x^*)$

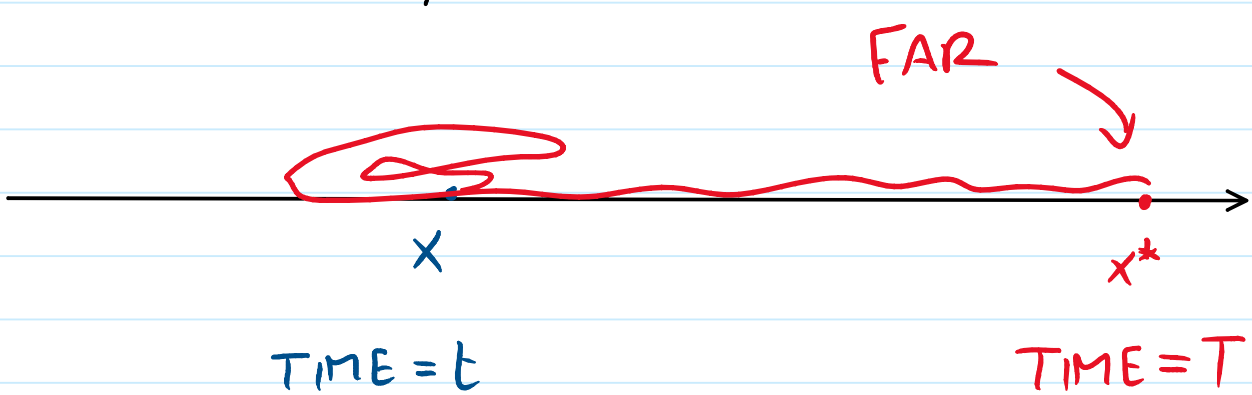
Let $\boxed{u(x,t) = \text{Average gain/loss starting at } x \text{ and } t}$

Then u solves $\begin{cases} u_t = u_{xx} & \text{in } R \times (0,T) \\ u(x,T) = g(x) & \text{on } R \end{cases}$



(This is called a TERMINAL value problem, as opposed to the INITIAL value problems that we're used to)

Note: Positivity here implies that if $g(x^*) > 0$ somewhere, then $u > 0$ everywhere, so it will always be possible to reach x^* at time T, no matter which time t you start at!



And with this we're officially done with the material of the course!!!

Congratulations, you made it!!!

The End