Welcome to our final exam review session! And of course, let's talk about the elephant in the room, separation of variables!!!

I- SEPARATION OF VARIABLES

You need to know how to separate variables for both the heat, wave, and Laplace's equation (with all different kinds of boundary conditions), but I might give you a completely different equation, like the following example

Example: Find a solution of

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u_{xx} - u & 0 < x < \pi \\
u(0,t) &= 0, \quad u(\pi,t) = 0 \\
u(x,1) &= 1
\end{align*}
\]

STEP 1: SEPARATION OF VARIABLES

\[u(x,t) = X(x) \ T(t)\]

Plug into \(\frac{\partial u}{\partial t} = u_{xx} - u\)

\[
\frac{\partial}{\partial t} (X(x) \ T(t)) = (X(x)T(t))_{xx} - X(x)T(t)
\]

\[
\frac{\partial}{\partial t} X(x) \ T'(t) = X''(x) \ T(t) - X(x) \ T(t)
\]

\[
\frac{\partial}{\partial t} T'(t) = \frac{X''(x)}{X(x)} \ T(t) - T(t)
\]
\[ \frac{\dot{T}'(t)}{T(t)} = \frac{X''(x)}{X(x)} - 1 \]

\[ X''(x) = \frac{\dot{T}'(t)}{T(t)} + 1 = \lambda \]

\[ \frac{X(x)}{T(t)} \]

**STEP 2: X EQUATION**

*SKIP* (same as before)

\[
\begin{align*}
X''(x) &= \lambda X(x) \\
X(0) &= 0 \\
X(\pi) &= 0
\end{align*}
\]

**STEP 3: 3 CASES**

*SKIP*

**Conclusion:** \[ \lambda = -m^2 \quad (m = 1, 2, ...) \]

\[ X(x) = \sin(mx) \quad (m = 1, 2, ...) \]

**STEP 4: T EQUATION**

\[ \frac{\dot{T}'(t)}{T(t)} + 1 = \frac{\lambda}{T(t)} = -m^2 \]

\[ \Rightarrow \frac{\dot{T}'}{T} = -m^2 - 1 \]

\[ \Rightarrow \frac{\dot{T}'}{T} = -(m^2+1) \]

\[ \Rightarrow \frac{\dot{T}'}{T} = -(m+1) \quad \dot{t} \]
\[ \Rightarrow (\ln|T|)' = -(m^2+1) \]

\[ \Rightarrow \ln|T| = -(m^2+1) \ln(t) + C \quad (t > 0) \]

\[ \Rightarrow |T| = e^{-\ln(t)(m^2+1) + C} = e^C t^{-(m^2+1)} \]

\[ \Rightarrow T(t) = +/- e^C t^{-(m^2+1)} \]

\[ \Rightarrow T(t) = C t^{-(m^2+1)} \quad (m = 1, 2, \ldots) \quad (X(x) = \sin(mx)) \]

**STEP 5: LINEAR COMBOS**

\[ u(x,t) = \sum_{M=1}^{\infty} A_m t^{-(m^2+1)} \sin(mx) \]

**STEP 6: u(x,1) = 1**

\[ u(x,1) = \sum_{M=1}^{\infty} A_m t^{-(m^2+1)} \sin(mx) = \sum_{M=1}^{\infty} A_m \sin(mx) = 1 \]

\[ A_m = \frac{2}{\pi} \int_{0}^{\pi} 1 \sin(mx) \, dx = \frac{2}{\pi} \left[ \frac{-\cos(mx)}{m} \right]_{0}^{\pi} \]

\[ = \frac{2}{\pi m} \left[ -\cos(\pi m) + 1 \right] = \frac{2}{\pi m} \left[ (-1)^{m^2+1} + 1 \right] \]

**STEP 7: ANSWER**

\[ u(x,t) = \sum_{M=1}^{\infty} \frac{2}{\pi m} \left[ (-1)^{m^2+1} + 1 \right] t^{-(m^2+1)} \sin(mx) \]

**II- FIRST-ORDER INTERLUDE**
Example: Solve \( g(y) u_x + h(x) u_y = 0 \)

\[
\frac{dy}{dx} = \text{Slope} = \frac{h(x)}{g(y)}
\]

\[
g(y) \, dy = h(x) \, dx
\]

\[
G(y) = H(x) + C
\]

\[
\Rightarrow C = G(y) - H(x)
\]

Solution: \( u(x,y) = f(G(y) - H(x)) \) where \( f \) is arbitrary

III- MAXIMUM PRINCIPLE AND ALL THAT

Example: Suppose \( u \) solves

\[
\begin{align*}
  u_t &= u_{xx} & 0 < x < 1 \\
  u(0,t) &= 0 \\
  u(1,t) &= 0 \\
  u(x,0) &= 4x(1-x)
\end{align*}
\]

Picture:

(a) Show \( 0 < u < 1 \) (for \( t > 0 \) and \( 0 < x < 1 \))
**MAXIMUM PRINCIPLE**

Max $u$ = the largest of: $\max u(0,t)$, $\max u(1,t)$, $\max u(x,0)$

= the largest of: 0, 0, and $\max 4x(1-x)$

$f(x) = 4x - 4x^2$, $f'(x) = 4 - 8x = 0 \Rightarrow x = 1/2$

$f(0) = 0$, $f(1) = 1$, $f(1/2) = 4/2(1/2) = 1$

So $\max u = \max 0, 0, 1$, which is 1

Hence $u \leq 1$ but by the strong max principle, $u < 1$

(Because if $u = 1$ inside, then $u$ attains its max inside, so $u$ must be constant)

Similarly

min $u$ = the smallest of 0, 0, $\min 4x(1-x)$

= the smallest of 0, 0, 0

= 0

$\Rightarrow u \geq 0$, and in fact $u > 0$ by strong max principle

So $0 < u < 1$

(b) Show $u(x,t) = u(1-x,t)$

Let $v(x,t) = u(1-x,t)$

One last time in this course: USE THE CHEN LU!!!
\[ v_x = (u(1-x,t))_x = -u_x(1-x,t) \]

\[ v_{xx} = u_{xx}(1-x,t) \]

So \( v_+ = v_{xx} \)

Moreover:

\[ v(0,t) = u(1-0,t) = u(1,t) = 0 \]
\[ v(1,t) = u(1-1,t) = u(0,t) = 0 \]

\[ v(x,0) = u(1-x,0) = 4(1-x)(1-(1-x)) = 4(1-x)(x) = 4x(1-x) \]

So \( v \) solves:

\[ \begin{cases} v_+ = v_{xx} \\ v(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = 4x(1-x) \end{cases} \]

Which is the same PDE as for \( u \) !!!

By uniqueness, \( v(x,t) = u(x,t) \), that is, \( u(1-x,t) = u(x,t) \)

( c ) Une energy methods to show that \( E(t) = \frac{1}{2} \int_0^1 u^2(x,t) \, dx \) is decreasing in \( t \)

**WARNING:** For the final I won't tell you which function to multiply your equation by any more!
Multiply $u_t = u_{xx}$ by $u$ and integrate

$$\int u_t \ u \ dx = \int u_{xx} \ u \ dx$$

$$A = B$$

$$A = \int 1/2 \ \frac{d}{dt} (u^2) \ dx = \frac{d}{dt} \left( \frac{1}{2} \int u^2 \ dx \right) = E'(t)$$

$$B = \text{IBP} = u_x(1,t) \ u(1,t) - u_x(0,t) \ u(0,t) - \int u_x \ u_x \ dx$$

$$= - \int (u_x)^2 \ dx$$

$A = B$ says: $E'(t) = - \int (u_x)^2 \ dx \leq 0$

$\Rightarrow E'(t) \leq 0$

$\Rightarrow E(t)$ is decreasing