LECTURE 28: FINAL EXAM REVIEW (II)

Wednesday, November 27, 2019 7:03 PM

Welcome to the second part of the final exam review session! Today we'll do more practice problems!

I- NUL(A), COL(A), RANK(A)

Example:

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \times 2$$

(a) Find a basis for Nul(A) $\qquad (Ax = 0)$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 5 & -10 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \times -5$$

$$\sim \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} & & \uparrow & & \uparrow & \\ & & z & & s & \end{array} \end{array}$$

$$\begin{cases} x + 3z = 0 \\ y - 3z - 7s = 0 \\ t - 2s = 0 \end{cases} \Rightarrow \begin{cases} x = -3z \\ y = 3z + 7s \\ t = 2s \end{cases}$$

$$x = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3z \\ 3z + 7s \\ z \\ 2s \\ s \end{bmatrix} = z \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) Find dim(Nul(A)) = 2 = # Free variables

(c) Find a basis for Col(A)

Pivots in 1st, 2nd, 4th columns

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) Find Rank(A) = dim(Col(A)) = 3 = # Pivots

(e) Is the Rank-Nullity Theorem true?

$$\text{Rank}(A) + \dim(\text{Nul}(A)) = n\ ?$$
$$3 \quad + \quad 2 \quad = 5\ ? \qquad \text{YES}$$

II- LEAST SQUARES

Example:

(a) Find the least-squares solution to $Ax = b$, where

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$Ax = b \Rightarrow A^T A\hat{x} = A^T b$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -11 & -4 \\ -11 & 22 & 11 \end{array}\right] \xrightarrow{(\div -11)} \left[\begin{array}{cc|c} 6 & -11 & -4 \\ 1 & -2 & -1 \end{array}\right] \xrightarrow{(\times -6)} \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & -2 & -1 \end{array}\right] \xrightarrow{(\times 2)}$$

$$\longrightarrow \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$$\hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(b) Find the Least-Squares Error

$$||A\hat{x} - b|| = \left|\left| \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right|\right|$$

$$= \left|\left| \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right|\right| = \sqrt{11}$$

III- $\mathscr{B}$-MATRIX

Example:

$$\mathscr{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

(a) Find the $\mathscr{B}$-matrix of $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$

Note: Fast Way: $B = P^{-1}AP$, $P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

$$A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3 & 2 & -6 & -1 \\ 2 & 1 & -4 & 0 \end{array}\right] \dashrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$$

P AP

$$\boxed{B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}}$$

(b) Find the coordinates of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with respect to $\mathcal{B}$

Recall: $[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$ means $x = a\begin{bmatrix} 3 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 1 & 1 \end{array}\right] \dashrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \dashrightarrow \boxed{[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

IV- APPLICATIONS OF DIAGONALIZATION

Example:

$$A = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$$

(a) Find A^n

Diagonalize A: $\lambda = 1 \dashrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 4 \dashrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = PDP^{-1}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$A^n = P\,D^n\,P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 4^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^n \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4^n & 4^n \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 4^n & -1 + 4^n \\ 2 - 2(4^n) & -1+2\,(4^n) \end{bmatrix}$$

(b) Calculate $\sqrt{A} = A^{1/2}$

$n = 1/2$

$$A^{1/2} = \begin{bmatrix} 2 - 4^{1/2} & -1 + 4^{1/2} \\ 2 - 2(4^{1/2}) & -1 + 2\,(4^{1/2}) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -1+2 \\ 2-2(2) & -1+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

What does that mean?

Note: $\underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}}_{\sqrt{A}} \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}}_{\sqrt{A}} = \underbrace{\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}}_{A}$ WOW!

V- CRAMER'S RULE

Example: Use Cramer's rule to solve $A\mathbf{x} = b$

$A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$x = \dfrac{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}} = \dfrac{-1}{-1} = 1$ $y = \dfrac{\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}} = \dfrac{1}{-1} = -1$

$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$