SOLUTIONS



1. *(5 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

F (a) The function $T : \mathbb{R}^{N} \to \mathbb{R}$ given by $T(u) = \|u\|$ is a linear transformation.

F (b) If $\{u, v, w\}$ is linearly dependent, then so is $\{u, v\}$

T (c) Any orthonormal set is linearly independent

T (d) If $A$ is square and $A^2$ is invertible, then $A$ is invertible

F (e) If $u$ and $v$ are (nonzero) eigenvectors of $A$, then $u + v$ is an eigenvector of $A$

EXPLANATIONS (OPTIONAL)

(a) $T(cU) = \|cU\| = |c|\,\|U\| \neq c\|U\| = cT(U)$

(b) $\left\{ \underset{U}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{V}{\begin{bmatrix}0\\1\end{bmatrix}}, \underset{W}{\begin{bmatrix}2\\0\end{bmatrix}} \right\}$ LD BUT $\left\{ \underset{U}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{V}{\begin{bmatrix}0\\1\end{bmatrix}} \right\}$ LI

(c) ANY ORTHOGONAL SET W/O 0 VECTOR IS LI
AND ORTHONORMAL SETS ARE ORTHOGONAL AND NONZERO
(BECAUSE THEY HAVE LENGTH 1)

(d) KNOW $DET(A^2) \neq 0$, SO $(DET(A))^2 \neq 0$ SO $DET(A) \neq 0$

(e) IF $AU = 2U$ AND $AV = 3V$ THEN
$A(U+V) = AU + AV = 2U + 3V \neq \lambda(U+V)$ !
(U AND V MUST CORRESPOND TO THE SAME EIGENVALUE FOR THIS TO WORK)

2. *(20 points)* Find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

1) $\underline{\text{EIGENVALUES}}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & -4 & -16 \\ -2 & \lambda-5 & -8 \\ 2 & 2 & \lambda+5 \end{vmatrix}$$

$$= (\lambda-7)\begin{vmatrix} \lambda-5 & -8 \\ 2 & \lambda+5 \end{vmatrix} + 4\begin{vmatrix} -2 & -8 \\ 2 & \lambda+5 \end{vmatrix} - 16\begin{vmatrix} -2 & \lambda-5 \\ 2 & 2 \end{vmatrix}$$

$$= (\lambda-7)\left[(\lambda-5)(\lambda+5)+16\right] + 4\left[-2(\lambda+5)+16\right] - 16\left[-4-2(\lambda-5)\right]$$

$$= (\lambda-7)(\lambda^2-25+16) + 4(-2\lambda-10+16) - 16(-4-2\lambda+10)$$

$$= (\lambda-7)(\lambda^2-9) + 4(-2\lambda+6) - 16(-2\lambda+6)$$

$$= (\lambda-7)\underbrace{(\lambda-3)}(\lambda+3) + 4(-2)\underbrace{(\lambda-3)} - 16(-2)\underbrace{(\lambda-3)}$$

$$= (\lambda-3)\left[(\lambda-7)(\lambda+3) - 8 + 32\right]$$

$$= (\lambda-3)\left[\lambda^2-4\lambda-21+24\right]$$

$$= (\lambda-3)(\lambda^2-4\lambda+3)$$

$$= (\lambda-3)(\lambda-3)(\lambda-1)$$

$$= (\lambda-3)^2(\lambda-1) = 0$$

$$\Longrightarrow \underline{\lambda=3} \quad \text{AND} \quad \underline{\lambda=1}$$

2) EIGENVECTORS

$\lambda = 3$ $\quad$ $\text{Nul}(3I - A)$

$$= \text{Nul} \begin{bmatrix} 3-7 & -4 & -16 \\ -2 & 3-5 & -8 \\ 2 & 2 & 3+5 \end{bmatrix}$$

$$= \text{Nul} \begin{bmatrix} -4 & -4 & -16 \\ -2 & -2 & -8 \\ 2 & 2 & 8 \end{bmatrix} \begin{matrix} (\div -4) \\ (\div -2) \\ (\div 2) \end{matrix}$$

$$= \text{Nul} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{bmatrix} \begin{matrix} (x-1) \\ (x-1) \end{matrix}$$

$$= \text{Nul} \begin{array}{c} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \quad\;\; \uparrow \;\; \uparrow \\ \quad\;\; y \;\; z \end{array}$$

$x + y + 4z = 0 \implies x = -y - 4z$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-4z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$ $\quad$ $\text{Nul}(I - A) = \text{Nul} \begin{bmatrix} 1-7 & -4 & -16 \\ -2 & 1-5 & -8 \\ 2 & 2 & 1+5 \end{bmatrix}$

$$= \text{NUL} \begin{bmatrix} -6 & -4 & -16 \\ -2 & -4 & -8 \\ 2 & 2 & 6 \end{bmatrix} \begin{array}{l} (\div -2) \\ (\div -2) \\ (\div 2) \end{array}$$

$$= \text{NUL} \begin{bmatrix} 3 & 2 & 8 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \text{NUL} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 8 \\ 1 & 1 & 3 \end{bmatrix} \begin{array}{l} (x-3) \\ (x-1) \end{array}$$

$$= \text{NUL} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & -1 & -1 \end{bmatrix} \begin{array}{l} \\ (\div -4) \\ (\div -1) \end{array}$$

$$= \text{NUL} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} (x-1)$$

$$= \text{NUL} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} (x-2) = \text{NUL} \begin{array}{c} \begin{array}{ccc} & & z \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{cases} x + 2z = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = -2z \\ y = -z \end{cases} \Rightarrow \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

3) <u>ANSWER</u>

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}}$$

3. *(5 points, 1 point each)* For the matrix $A$ below, find the following. Justify your answer.

(a) $\dim(Nul(A))$
(b) $Rank(A)$
(c) A basis for $Col(A)$
(d) State the Rank Theorem
(e) A basis for $(Nul(A^T))^\perp$

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim B = \begin{array}{cccc} & & z & t \\ & & \downarrow & \downarrow \\ \multicolumn{4}{c}{\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \end{array}$$

(a) $\text{DIM}(\text{NUL}(A)) = \#$ FREE VARIABLES $= 2$

(b) $\text{RANK}(A) = \#$ PIVOTS $= 2$

(c) PIVOTS IN COLUMNS 1 AND 2

BASIS FOR $\text{COL}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$

(d) $\text{DIM}(\text{NUL}(A)) + \text{RANK}(A) = N = 4$

$(2 + 2 = 4)$

(e) $(\text{NUL}(A^T))^{\perp} = \text{COL}((A^T)^T) = \text{COL}(A)$ (PUT THAT FROWN UPSIDE-DOWN)

BASIS $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$

4. *(10 points)* Find the equation of the line $y = ax + b$ that best fits the points $(-1, 0), (0, 1), (1, 2), (2, \cancel{4})$ in a least-squares sense.

5

1) SUPPOSE THE POINTS ARE ON THE LINE $y = ax + b$

THEN:

$$a(-1) + b = 0$$
$$a(0) + b = 1$$
$$a(1) + b = 2$$
$$a(2) + b = 5$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}}_{b}$$

2) $A^T A \hat{x} = A^T b$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$\begin{matrix} (\div 2) \\ (\div 2) \end{matrix} \left[\begin{array}{cc|c} 6 & 2 & 12 \\ 2 & 4 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 6 \\ 1 & 2 & 4 \end{array}\right] \uparrow (\times -3) \to \left[\begin{array}{cc|c} 0 & -5 & -6 \\ 1 & 2 & 4 \end{array}\right] (\div -5)$$

$$\to \left[\begin{array}{cc|c} 0 & 1 & 6/5 \\ 1 & 2 & 4 \end{array}\right] \downarrow (\times -2) \to \left[\begin{array}{cc|c} 0 & 1 & 6/5 \\ 1 & 0 & 8/5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 8/5 \\ 0 & 1 & 6/5 \end{array}\right]$$

3) ANSWER: $y = ax + b = \boxed{\frac{8}{5}x + \frac{6}{5}}$

SOMETIMES



5. *(5 points, 1 point each)* Write **ALWAYS** if $A$ is always diagonalizable, [redacted] if $A$ might or might not be diagonalizable, and **NEVER** if $A$ is never diagonalizable. Here $A$ is a $3 \times 3$ matrix, and you do **not** have to justify your answer.

SOMETIMES (a) $A$ is invertible

SOMETIMES (b) $A$ only has eigenvalue $\lambda = 2$ ( $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ IS BUT $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ISN'T )

ALWAYS (c) $A$ has eigenvalues $\lambda = 2, 0, -4$ ( 3 DISTINCT EIGENVALUE )

ALWAYS (d) $A$ has eigenvalues $\lambda = 1, 2$ and $\dim(E_1) = 2$, $\dim(E_2) = 1$

( 3 LI EIGENVECTORS )

NEVER (e) $A$ has eigenvalues $\lambda = 3, 4$ and $\dim(E_3) = 1$, $\dim(E_4) = 1$.

( ONLY 2 LI EIGENVECTORS )

6. *(20 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer.

TRUE

(a) If $Q$ is orthogonal, then $\|Qx\| = \|x\|$ for all $x$

$$\|Qx\|^2 = (Qx)\cdot(Qx)$$
$$= (Qx)^T(Qx)$$
$$= x^T \underbrace{Q^T Q}_{I} x$$
$$= x^T x$$
$$= \|x\|^2$$

So $\|Qx\|^2 = \|x\|^2$

$\Rightarrow \|Qx\| = \|x\|$

TRUE

(b) If $A$ is invertible and $\mathbf{v}$ is a nonzero eigenvector of $A$, then $\mathbf{v}$ is also an eigenvector of $A^{-1}$ (FOR WHICH EIGENVALUE ?)

SUPPOSE $AV = \lambda V$

$$\Rightarrow A^{-1}(AV) = A^{-1}(\lambda V)$$
$$\Rightarrow IV = \lambda A^{-1}V$$
$$\Rightarrow \frac{V}{\lambda} = \frac{\lambda A^{-1}V}{\lambda}$$
$$\Rightarrow A^{-1}V = \left(\frac{1}{\lambda}\right)V$$

SO $V$ IS STILL AN EIGENVECTOR,

BUT FOR THE EIGENVALUE $\frac{1}{\lambda}$

TRUE

(c) **Definition:** $A$ is positive-definite if $x^T Ax > 0$ for any $x \neq 0$

If $A$ is positive definite, then any eigenvalue of $A$ (if it exists) must be positive.

SUPPOSE $AV = \lambda V$ WITH $V \neq 0$

BY DEFINITION WITH $x = V$, GET

$$(V^T AV) > 0 \Rightarrow V^T (\lambda V) > 0$$
$$\Rightarrow \lambda V^T V > 0$$
$$\Rightarrow \lambda \underbrace{\|V\|^2}_{> 0 \text{ (SINCE } V \neq 0)} > 0 \Rightarrow \lambda > 0$$

TRUE

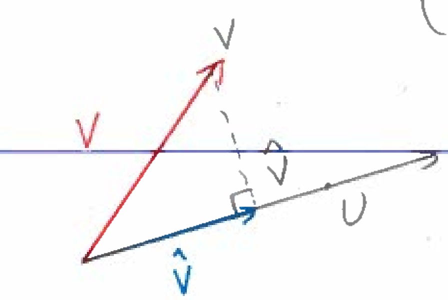
(d) For any vectors $u$ and $v$ in $\mathbb{R}^n$, we have

$$|u \cdot v| \leq \|u\| \, \|v\|$$

**Hint:** Calculate the orthogonal projection ~~$\hat{u}$ of $u$ on Span$\{v\}$~~ $\hat{V}$ OF $V$ ON SPAN$\{U\}$ and compare [redacted] and [redacted]. A picture helps.

$\|\hat{V}\|$ $\|V\|$

FIRST OF ALL, $\hat{V} = \left(\frac{V \cdot U}{U \cdot U}\right) U$




NOW $\|\hat{V}\| \leq \|V\|$

$$\Rightarrow \left\| \left(\frac{V \cdot U}{U \cdot U}\right) U \right\| \leq \|V\|$$

$$\Rightarrow \left| \frac{V \cdot U}{U \cdot U} \right| \|U\| \leq \|V\| \Rightarrow \frac{|V \cdot U|}{\|U\|^{\cancel{2}}} \cancel{\|U\|} \leq \|V\|$$

$$\Rightarrow |U \cdot V| \leq \|U\| \, \|V\|$$

7. *(10 points)* The awesome thing is that everything that we've been doing works not only for vectors in $\mathbb{R}^n$, but also for functions!

**Definition:** If $f$ and $g$ are functions, then

$$f \cdot g = \int_{-1}^{1} f(x)g(x)dx$$

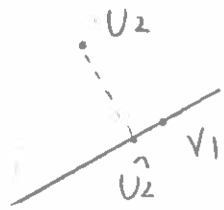
$U_1 \; U_2 \; U_3$

Use the Gram-Schmidt process to find an orthogonal basis of Span $\{\cancel{1}, \cancel{x}, x^2\}$

$\{1, x, x^2\}$.

**Hint:** Just use the usual Gram-Schmidt process with $u_1 = 1$, $u_2 = x$ and $u_3 = x^2$ and the dot product above!

GOAL FIND $\{V_1, V_2, V_3\}$ $\perp$

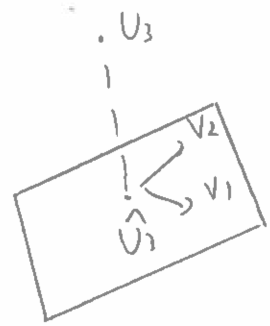



$$V_1 = U_1 = 1$$

$$\hat{U_2} = \left(\frac{U_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1$$

$$= \left(\frac{\int_{-1}^{1} x(1)\,dx}{\int_{-1}^{1}(1)(1)\,dx}\right) 1 = \left(\frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx}\right) 1 = \left(\frac{\left[\frac{x^2}{2}\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}}\right) 1 = 0$$

$$V_2 = U_2 - \hat{U_2} = x - 0 = x$$




$$\hat{U_3} = \left(\frac{U_3 \cdot U_1}{U_1 \cdot U_1}\right) V_1 + \left(\frac{U_3 \cdot U_2}{U_2 \cdot U_2}\right) V_2$$

$$= \left(\frac{\int_{-1}^{1} x^2\, 1\,dx}{\int_{-1}^{1} 1\, 1\,dx}\right) 1 + \left(\frac{\int_{-1}^{1} x^2 x\,dx}{\int_{-1}^{1} x\, x\,dx}\right) x$$

$$= \left( \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} \right) 1 + \left( \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} \right) x$$

$$= \left( \frac{\left[\frac{x^3}{3}\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}} \right) + \left( \frac{\left[\frac{x^4}{4}\right]_{-1}^{1}}{\left[\frac{x^3}{3}\right]_{-1}^{1}} \right) x$$

$$= \left( \frac{\frac{1}{3} + \frac{1}{3}}{1+1} \right) + \left( \frac{\cancel{\frac{1}{4}} - \cancel{\frac{1}{4}}}{\frac{1}{3} + \frac{1}{3}} \right) x$$

$$= \frac{\frac{2}{3}}{2} + 0 = \frac{1}{3}$$

$$V_3 = U_3 - \hat{U}_3 = x^2 - \frac{1}{3}$$

ANSWER $\boxed{\{V_1, V_2, V_3\} = \{1, x, x^2 - \frac{1}{3}\}}$

NOTE THESE POLYNOMIALS ARE SOMETIMES CALLED THE LEGENDRE POLYNOMIALS AND ARE VERY USEFUL FOR COMPUTING

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8. *(10 points)* Calculate

$$\tan^{-1}\left(\begin{bmatrix}-1 & 2\\ -1 & 2\end{bmatrix}\right)$$

**Hint:** This is the same thing that we've been doing in lecture for $A^n$ and $\sqrt{A}$, except here you apply $\tan^{-1}$ to all the diagonal entries of $D$. See footnote below[1]

1) DIAGONALIZE $\quad A = \begin{bmatrix}-1 & 2\\ -1 & 2\end{bmatrix}$

EIGENVALUES $\quad \text{DET}(\lambda I - A) = \begin{vmatrix}\lambda+1 & -2\\ 1 & \lambda-2\end{vmatrix}$

$= (\lambda+1)(\lambda-2)+2 = \lambda^2 - \lambda \cancel{-2} \cancel{+2} = \lambda(\lambda-1) = 0$

$\Rightarrow \underline{\lambda = 0, 1}$

EIGENVECTORS $\quad \underline{\lambda = 0} \quad \text{NUL}(0I - A) = \text{NUL}\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix} \downarrow (\times -1)$

$= \text{NUL}\begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix} \qquad x - 2y = 0 \Rightarrow x = 2y \Rightarrow \underline{x} = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2y\\ y\end{bmatrix} = y\begin{bmatrix}2\\ 1\end{bmatrix}$

$\underline{\lambda = 1} \quad \text{NUL}(I - A) = \text{NUL}\begin{bmatrix}1+1 & -2\\ 1 & 1-2\end{bmatrix} = \text{NUL}\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix} = \text{NUL}\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$

$x - y = 0 \Rightarrow x = y \Rightarrow \underline{x} = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}y\\ y\end{bmatrix} = y\begin{bmatrix}1\\ 1\end{bmatrix}$

SO $\quad A = PDP^{-1} \quad$ WITH $\quad D = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad P = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$

2) BY HINT, $\quad \text{TAN}^{-1}(A) = P\,\text{TAN}^{-1}(D)\,P^{-1}$

[1]Here are some useful (and useless) values of arctan: $\tan^{-1}(0) = 0, \tan^{-1}(\pm 1) = \pm\frac{\pi}{4}, \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}, \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}, \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$= \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\text{TAN}^{-1}(0) & 0\\ 0 & \text{TAN}^{-1}(1)\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}^{-1}$

$= \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \pi/4\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\ -\pi/4 & \pi/2\end{bmatrix} = \boxed{\begin{bmatrix}-\pi/4 & \pi/2\\ -\pi/4 & \pi/2\end{bmatrix}}$ WHOA!

$$= (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & z+y+x \\ 1 & t+y+x \end{vmatrix}$$

$$= (y-x)(z-x)(t-x)(z-y)(t-y)\left(t+\cancel{y}+\cancel{x}-z-\cancel{y}-\cancel{x}\right)$$

$$= (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$$



9. *(15 = 8 + 7 points)*

(a) Calculate the determinant of the following Vandermonde matrix. Write your answer in factored form. Here $x, y, z, t$ are numbers.

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix}$$

**Hint:** $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$\begin{array}{c} \\ (x-1) \\ (x-1) \\ (x-1) \end{array}\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{1} & \cancel{x} & \cancel{x^2} & \cancel{x^3} \\ 0 & y-x & (y-x)(y+x) & (y-x)(y^2+xy+x^2) \\ 0 & z-x & (z-x)(z+x) & (z-x)(z^2+xz+x^2) \\ 0 & t-x & (t-x)(t+x) & (t-x)(t^2+xt+x^2) \end{vmatrix}$$

$$= (y-x)(z-x)(t-x)\begin{vmatrix} 1 & y+x & y^2+xy+x^2 \\ 1 & z+x & z^2+xz+x^2 \\ 1 & t+x & t^2+xt+x^2 \end{vmatrix}\begin{array}{c} \\ (x-1) \\ (x-1) \end{array}$$

$$= (y-x)(z-x)(t-x)\begin{vmatrix} \cancel{1} & \cancel{y+x} & \cancel{y^2+xy+x^2} \\ 0 & (z+\cancel{x})-(y+\cancel{x}) & z^2+xz+\cancel{x^2}-y^2-yx-\cancel{x^2} \\ 0 & (t+\cancel{x})-(y+\cancel{x}) & t^2+xt+\cancel{x^2}-y^2-yx-\cancel{x^2} \end{vmatrix}$$

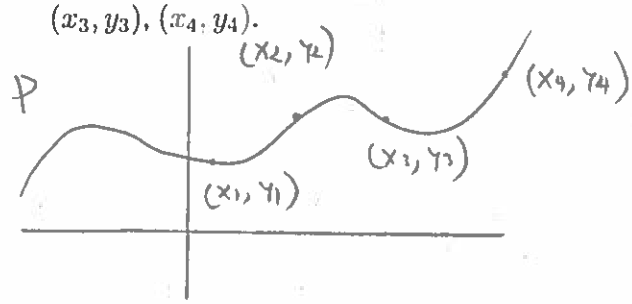
$$= (y-x)(z-x)(t-x)\begin{vmatrix} z-y & z^2-y^2+xz-yx \\ t-y & t^2-y^2+xt-yx \end{vmatrix}$$

$$= (y-x)(z-x)(t-x)\begin{vmatrix} z-y & (z-y)(z+y)+x(z-y) \\ t-y & (t-y)(t+y)+x(t-y) \end{vmatrix}$$

( SEE ABOVE )

$$P(x) = a + bx + cx^2 + dx^3$$

(b) Let $x_1, x_2, x_3, x_4$ be distinct numbers and $y_1, y_2, y_3, y_4$ be any numbers. Use your result in $(a)$ to show that there exists exactly one cubic polynomial ~~$P(x) = a + bx + cx^2 + dx^3$~~ (for some $a, b, c, d$) that goes through the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$.




BY ASSUMPTION $\quad P(x_1) = y_1, \; P(x_2) = y_2, \; P(x_3) = y_3, \; P(x_4) = y_4$

$$a + bx_1 + cx_1^2 + dx_1^3 = y_1$$
$$a + bx_2 + cx_2^2 + dx_2^3 = y_2$$
$$a + bx_3 + cx_3^2 + dx_3^3 = y_3$$
$$a + bx_4 + cx_4^2 + dx_4^3 = y_4$$

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\underline{b}}$$

BUT $\quad \det(A) = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix}$ $\quad$ (a) WITH $x = x_1$, $y = x_2$, $z = x_3$, $t = x_4$

$$= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$$

$\neq 0$ SINCE THE $x_i$ ARE DISTINCT, SO $\det(A) \neq 0$

(TURN PAGE)

HENCE $A$ IS INVERTIBLE

SO $A\underline{x} = \underline{b}$ HAS A UNIQUE SOLUTION $\underline{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

WHICH MEANS THERE IS A UNIQUE POLYNOMIAL $P = a + bx + cx^2 + dx^3$

WHICH GOES THROUGH THE GIVEN POINTS.

NOTE THIS WORKS EVEN IF THE $y_i$ ARE NOT DISTINCT!

CHECK OUT THE "LAGRANGE INTERPOLATION FORMULA"

IF YOU WANT TO SEE HOW TO CONSTRUCT $P$.