MATH 3A – FINAL EXAM

Name: ________________________

Student ID: ________________________

Instructions: This is it, your final hurdle to freedom! Welcome to your Final Exam! You have 120 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. **You will lose 1 point if you don't fill out all the info on this page.** May your luck be orthogonal! :)

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: __________________________________

1		5
2		20
3		5
4		10
5		5
6		20
7		10
8		10
9		15
Total		100

Date: Friday, December 13, 2019.

1. *(5 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

   (a) The function $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(u) = \|u\|$ is a linear transformation.

   (b) If $\{u, v, w\}$ is linearly dependent, then so is $\{u, v\}$

   (c) Any orthonormal set is linearly independent

   (d) If A is square and A^2 is invertible, then A is invertible

   (e) If u and v are (nonzero) eigenvectors of A, then $u + v$ is also an eigenvector of A

2. *(20 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

3. *(5 points, 1 point each)* For the matrix A below, find the following. Justify your answer.

   (a) $\dim(Nul(A))$
   (b) $Rank(A)$
   (c) A basis for $Col(A)$
   (d) State the Rank Theorem
   (e) A basis for $(Nul(A^T))^\perp$

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. *(10 points)* Find the equation of the line $y = ax + b$ that best fits the points $(-1, 0), (0, 1), (1, 2), (2, 5)$ in a least-squares sense.

5. *(5 points, 1 point each)* Write **ALWAYS** if A is always diagonalizable, **SOMETIMES** if A might or might not be diagonalizable, and **NEVER** if A is never diagonalizable. Here A is a 3×3 matrix, and you do **not** have to justify your answer.

(a) A is invertible

(b) A only has eigenvalue $\lambda = 2$

(c) A has eigenvalues $\lambda = 2, 0, -4$

(d) A has eigenvalues $\lambda = 1, 2$ and $\dim(E_1) = 2$, $\dim(E_2) = 1$

(e) A has eigenvalues $\lambda = 3, 4$ and $\dim(E_3) = 1$, $\dim(E_4) = 1$.

6. *(20 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer.

   (a) If Q is orthogonal, then $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$

   (b) If A is invertible and $\mathbf{v}$ is a nonzero eigenvector of A, then $\mathbf{v}$ is also an eigenvector of A^{-1}

(c) **Definition:** A is positive-definite if $\mathbf{x}^T A\mathbf{x} > 0$ for any $\mathbf{x} \neq 0$

If A is positive definite, then any eigenvalue of A (if it exists) must be positive.

(d) For any vectors u and v in $\mathbb{R}^n$, we have

$$|u \cdot v| \leq \|u\| \, \|v\|$$

Hint: Calculate the orthogonal projection $\hat{v}$ of v on Span$\{u\}$ and compare $\|\hat{v}\|$ and $\|v\|$. A picture helps.

7. *(10 points)* The awesome thing is that everything that we've been doing works not only for vectors in $\mathbb{R}^n$, but also for functions!

Definition: If f and g are functions, then

$$f \cdot g = \int_{-1}^{1} f(x)g(x)dx$$

Use the Gram-Schmidt process to find an orthogonal basis of

$$W = Span\left\{1, x, x^2\right\}$$

Hint: Just use the usual Gram-Schmidt process with $u_1 = 1$, $u_2 = x$ and $u_3 = x^2$ and the dot product above!

8. *(10 points)* Calculate

$$\tan^{-1}\left(\begin{bmatrix}-1 & 2\\ -1 & 2\end{bmatrix}\right)$$

Hint: This is the same thing that we've been doing in lecture for A^n and $\sqrt{A}$, except here you apply $\tan^{-1}$ to all the diagonal entries of D. Also see footnote below[1]

[1] Here are some useful (and useless) values of arctan: $\tan^{-1}(0) = 0, \tan^{-1}(\pm 1) = \pm\frac{\pi}{4}, \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}, \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}, \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

9. *(15 = 8 + 7 points)*

(a) Calculate the determinant of the following Vandermonde matrix. Write your answer in factored form. Here x, y, z, t are numbers.

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix}$$

Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(b) Let x_1, x_2, x_3, x_4 be distinct numbers and y_1, y_2, y_3, y_4 be any numbers. Use your result in (a) to show that there exists a unique cubic polynomial $P(x) = a + bx + cx^2 + dx^3$ (for some a, b, c, d) that goes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).