

# SOLUTIONS

1. (5 points, 1 point each) Circle the correct answer or fill in the blanks.  
No justification required, except for (c).

(a) The PDE  $2u_{xx} + (x^2)u_{xy} + (e^y)u = \sin(x)$  is

LINEAR

NOT LINEAR

(LINEAR IN U)

(b) The PDE  $2u_{xx} + 5u_y = 3u$  is:  $\Rightarrow 2U_{xx} + 5U_y - 3U = 0$

HOMOGENEOUS

NOT HOMOGENEOUS

(c) The type of the second-order PDE  $2u_{xx} + 3u_{xy} + u_{yy} + 6u_x + 8u_y - u = 0$  is

HYPERBOLIC because:

$$\underline{3^2 - 4(2)(1)} = 9 - 8 = 1$$

(d) Write out your favorite PDE that is neither first-order nor the Laplace/heat/wave equation

$$\underline{\text{I LIKE } U_{xx} U_{yy} - (U_{xy})^2 = f(x,y)}$$

(MONGE-AMPE`RE EQUATION)

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2. (25 points) Find a solution to the following PDE. Here  $0 < x < 1$  and  $0 < y < \pi$

$$\begin{cases} (x^2) u_{xx} + (x) u_x + u_{yy} = 0 \\ u(x, 0) = 0 \quad u(x, \pi) = 0 \\ \lim_{x \rightarrow 0^+} u(x, y) = 0 \quad u(1, y) = y \end{cases}$$

**Hint:** At some point, you'll have to solve a strange ODE. For this ODE, guess that your solution is of the form  $x^\alpha$  for some  $\alpha$  (or  $y^\alpha$  if you're dealing with  $Y$ ) AND THEN TAKE LINEAR COMBOS

STEP 1

SEPARATION OF VARIABLES

$$u(x, y) = X(x) Y(y)$$

$$\text{PLUG INTO PDE: } x^2 (X Y)_{xx} + x (X Y)_x + (X Y)_{yy} = 0$$

$$x^2 X'' \underline{Y} + x X' \underline{Y} + XY'' = 0$$

$$(x^2 X'' + x X') Y = -XY''$$

$$\Rightarrow \frac{Y''}{Y} = -\frac{(x^2 X'' + x X')}{X} = \lambda$$

STEP 2

$Y$ -EQUATION

$$\frac{Y''}{Y} = \lambda \Rightarrow Y'' = \lambda Y$$

$$u(x, 0) = 0 \Rightarrow \cancel{X(x)} Y(0) = 0 \Rightarrow Y(0) = 0$$

$$u(x, \pi) = 0 \Rightarrow \cancel{X(x)} Y(\pi) = 0 \Rightarrow Y(\pi) = 0$$

$$\Rightarrow \begin{cases} Y'' = \lambda Y \\ Y(0) = 0 \\ Y(\pi) = 0 \end{cases}$$

STEP 3

3 CASES

$$\text{CASE 1 } \lambda > 0, \text{ THEN } \lambda = \omega^2 \text{ FOR } \omega > 0$$

$$\text{so } Y'' = \lambda Y \Rightarrow Y'' = \omega^2 Y$$

$$\text{AUX } \Gamma' = \omega^2 \Rightarrow \Gamma = \pm \omega$$

$$\Rightarrow Y(Y) = Ae^{\omega Y} + Be^{-\omega Y}$$

$$Y(0) = Ae^{\omega 0} + Be^{-\omega 0} = A + B = 0$$

$$\Rightarrow B = -A \Rightarrow Y(Y) = Ae^{\omega Y} - Ae^{-\omega Y}$$

$$Y(\pi) = Ae^{\omega \pi} - Ae^{-\omega \pi} = 0$$

$$\Rightarrow Ae^{\omega \pi} = Ae^{-\omega \pi}$$

$$\Rightarrow e^{\omega \pi} = e^{-\omega \pi}$$

$$\Rightarrow \omega \pi = -\omega \pi$$

$$\Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

$$\text{BUT THEN } \lambda = \omega^2 = 0 \Rightarrow \lambda = 0$$

$$\text{CASE 2 } \lambda = 0$$

$$\text{THEN } Y'' = 0 \Rightarrow Y(Y) = Ay + B$$

$$Y(0) = A0 + B = B = 0$$

$$\Rightarrow Y(Y) = Ay \quad (\text{AYYY, LMNO!})$$

$$Y(\pi) = A\pi = 0 \Rightarrow A = 0$$

$$\Rightarrow Y(Y) = 0Y = 0 \Rightarrow \Leftarrow$$

$$\text{CASE 3 } \lambda < 0 \text{ THEN } \lambda = -\omega^2 \quad (\omega > 0)$$

$$\text{so } Y'' = \lambda Y \Rightarrow Y'' = -\omega^2 Y$$

$$\text{AUX } \Gamma' = -\omega^2 \Rightarrow \Gamma = \pm \omega i$$

$$\Rightarrow Y(Y) = A \cos(\omega Y) + B \sin(\omega Y)$$

$$Y(0) = A \cos(0) + B \sin(0) = A = 0$$

$$\Rightarrow Y(Y) = B \sin(\omega Y)$$

$$Y(\pi) = B \sin(\omega \pi) = 0$$

$$\Rightarrow \sin(\omega \pi) = 0$$

$$\Rightarrow \omega \pi = \pi M \quad (M=1, 2, \dots)$$

$$\Rightarrow \omega = M$$

CONCLUSION

$$\lambda = -M^2 \quad (M=1, 2, \dots)$$

$$Y(Y) = \sin(\omega Y) = \sin(MY)$$

STEP 4 X-EQUATION

$$+ \left( \frac{x' X'' + x X'}{X} \right) = \lambda = -M^2$$

$$\Rightarrow x' X'' + x X' = M^2 X$$

$$\Rightarrow x' X'' + x X' - M^2 X = 0$$

BY THE HWT, GUESS  $X(x) = x^\alpha$

$$\Rightarrow x^\alpha \alpha(\alpha-1)x^{\alpha-2} + x \alpha x^{\alpha-1} - M^2 x^\alpha = 0$$

$$\Rightarrow \alpha(\alpha-1)x^\alpha + \alpha x^\alpha - M^2 x^\alpha = 0$$

$$\Rightarrow \alpha^2 - \alpha - M^2 = 0$$

$$\Rightarrow \alpha^2 = M^2 \Rightarrow \alpha = \pm M$$

SAYS  $X(x) = x^M$  AND  $x^{-M}$  solve

THE X-EQUATION, AND SO

$$X(x) = A_n x^M + B_n x^{-M}$$

STEP 5

LINEAR COMBOS

$$U(x, y) = \sum_{M=1}^{\infty} (A_M x^M + B_M x^{-M}) \sin(My)$$

STEP 6

$$\lim_{x \rightarrow 0^+} U(x, y) = 0$$

$$\lim_{x \rightarrow 0^+} U(x, y) = \sum_{M=1}^{\infty} \left( A_M \underbrace{\lim_{x \rightarrow 0^+} x^M}_{0} + B_M \underbrace{\lim_{x \rightarrow 0^+} x^{-M}}_0 \right) \sin(My)$$

BUT  $\lim_{x \rightarrow 0^+} x^{-M} = \infty$ , so UNLESS ALL THE  $B_M$  ARE 0,

THE SOLUTION BLOWS UP!

CONCLUSION  $B_M = 0$  FOR ALL M

$$\Rightarrow U(x, y) = \sum_{M=1}^{\infty} A_M x^M \sin(My)$$

STEP 7

$$U(1, y) = y$$

$$U(1, y) = \sum_{M=1}^{\infty} A_M 1^M \sin(My) = \sum_{M=1}^{\infty} A_M \sin(My) = y$$

$$\begin{matrix} + & y \leftrightarrow \sin(My) \\ - & 1 \leftrightarrow -\cos(My)/M \\ 0 & -\sin(My)/M^2 \end{matrix}$$

$$A_M = \frac{2}{\pi} \int_0^{\pi} y \sin(My) dy = \frac{2}{\pi} \left[ -y \frac{\cos(My)}{M} + \frac{\sin(My)}{M^2} \right]_0^{\pi} = -\frac{2}{\pi} \pi \frac{\cos(\pi M)}{M} = \frac{-2(-1)^M}{M} = \frac{2(-1)^{M+1}}{M}$$

STEP 8

CONCLUSION

$$U(x, y) = \sum_{M=1}^{\infty} \frac{2(-1)^{M+1}}{M} x^M \sin(My)$$

3. (20 points) Solve the following PDE:

$$\left\{ \begin{array}{l} \cancel{4U_{tt} - 5U_{xt} + U_{xx} = 0} \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{array} \right.$$

Note: Derive everything from scratch. The *only* thing you're allowed to assume is how to solve first-order PDEs.

$$\begin{aligned} & \left( \frac{\partial^2}{\partial t^2} - 5 \frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial x^2} \right) U = 0 \quad 4t^2 - 5xt + x^2 \\ & \Rightarrow \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \underbrace{\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)}_V U = 0 \\ & \Rightarrow \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) V = 0 \end{aligned}$$

$$\Rightarrow 4V_t - V_x = 0$$

$$\Rightarrow V(x, t) = f(4x + t) \quad (\text{f ARBITRARY})$$

$$\Rightarrow U_t - U_x = f(4x + t)$$

$$\text{Now } U_t - U_x = 0 \Rightarrow U(x, t) = G(x + t) \quad (G \text{ ARBITRARY})$$

For THE PARTICULAR SOLUTION OF  $U_t - U_x = f(4x + t)$ ,

$$\text{GUESS } U(x, t) = \alpha F(4x + t)$$

$$(\alpha F(4x + t))_t - (\alpha F(4x + t))_x = f(4x + t)$$

$$\alpha F'(4x + 4t) - 4\alpha F'(4x + t) = f(4x + t)$$

$$-3\alpha f(x + 4t) = f(x + 4t) \Rightarrow -3\alpha = 1 \Rightarrow \alpha = -\frac{1}{3}$$

SOLUTION  $U(x, t) = -\frac{1}{3} F(4x + t) + G(x + t)$

$$U(x, t) = F(4x + t) + G(x + t) \quad (F \text{ AND } G \text{ ARBITRARY})$$

$$2) \quad u(x,0) = F(4x+0) + G(x+0) = \phi(x)$$

$$\underline{F(4x)+G(x)} = \phi(x)$$

$$\begin{aligned} u_t(x,t) &= (F(4x+t))_t + (G(x+t))_t \\ &= F'(4x+t) + G'(x+t) \end{aligned}$$

$$u_t(x,0) = F'(4x) + G'(x) = \psi(x)$$

$$\Rightarrow F'(4x) + G'(x) = \psi(x)$$

$$\Rightarrow \int_0^t F'(4s) + G'(s) ds = \int_0^t \psi(s) ds$$

$$\Rightarrow \left[ \frac{1}{4} F(4s) + G(s) \right]_0^X = \int_0^X \psi(s) ds$$

$$\Rightarrow \frac{1}{4} F(4X) + G(X) - \underbrace{\frac{1}{4} F(0) - G(0)}_A = \int_0^X \psi(s) ds$$

HENCE

$$(x-\frac{1}{4}) \downarrow \left\{ \begin{array}{l} F(4x) + G(x) = \phi(x) \\ \frac{1}{4} F(4x) + G(x) = \int_0^x \psi(s) ds + A \end{array} \right.$$

$$\text{SUBTRACT} \quad \frac{1}{4} F(4x) = \phi(x) - \int_0^x \psi(s) ds - A$$

$$F(4x) = \frac{4}{3} \phi(x) + \frac{4}{3} \int_x^0 \psi(s) ds - \frac{4}{3} A \Rightarrow F(x) = \frac{4}{3} \phi(\frac{x}{4})$$

" ADD "

$$\frac{3}{4} G(x) = -\frac{\phi(x)}{4} + \int_0^x \psi(s) ds + A$$

$$\Rightarrow G(x) = -\frac{1}{3} \phi(x) + \frac{4}{3} \int_0^x \psi(s) ds + \frac{4}{3} A$$

3) ANSWER  $u(x,t) = F(4x+t) + G(x+t)$

$$= \frac{4}{3} \phi\left(\frac{4x+t}{4}\right) + \frac{4}{3} \int_{(4x+t)/4}^0 \psi(s) ds - \cancel{\frac{4}{3} A} - \frac{1}{3} \phi(x+t) + \frac{4}{3} \int_0^{x+t} \psi(s) ds + \cancel{\frac{4}{3} A}$$

4. (5 points) Find a solution of the following heat equation on the half-line (here  $x > 0$ )

$$\begin{cases} u_t = ku_{xx} \\ u_x(0, t) = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

Write your solution in terms of one integral. No need to write out the explicit formula for  $S(x, t)$  and no need to check that your solution works.

LET  $\Phi_{\text{EVEN}}(x) = \begin{cases} \phi(x) & \text{IF } x > 0 \\ \phi(-x) & \text{IF } x < 0 \end{cases}$

SOLVE  $\begin{cases} U_t = kU_{xx} & -\infty < x < \infty \\ U(x, 0) = \Phi_{\text{EVEN}}(x) \end{cases}$

$$U(x, t) = S(x, t) * \Phi_{\text{EVEN}}(x)$$

$$= \int_{-\infty}^{\infty} S(x-y, t) \Phi_{\text{EVEN}}(y) dy$$

$$\begin{aligned} p = -y &= \int_{-\infty}^0 S(x-y, t) \Phi(-y) dy + \int_0^{\infty} S(x-y, t) \Phi(y) dy \\ dp = -dy &\quad \downarrow \\ p(0) = 0 &= \int_{\infty}^0 S(x+p, t) \Phi(p) (-dp) + \int_0^{\infty} S(x-y, t) \Phi(y) dy \\ p(-\infty) = \infty &= \int_0^{\infty} S(x+y, t) \Phi(y) dy + \int_0^{\infty} S(x-y, t) \Phi(y) dy \end{aligned}$$

$$= \int_c^{\infty} [S(x+y, t) + S(x-y, t)] \Phi(y) dy$$

5. (10 points) Find the Fourier sine series of  $f(x) = \cos(x)$  on  $(0, \pi)$ . SIMPLIFY AS MUCH AS THERE IS A CONTRADICTION WHEN YOU PLUG IN  $x = 0$ ? EXPLAIN AS YOU CAN!

**Hint:**

$$\cos(A)\sin(B) = \frac{1}{2}[\sin(B+A) + \sin(B-A)]$$

$$\cos(x) = \sum_{M=1}^{\infty} B_M \sin(Mx)$$

$$B_M = \frac{2}{\pi} \int_0^\pi \cos(x) \sin(Mx) dx$$

$$= \frac{2}{\pi} \int_0^\pi \frac{1}{2} [\sin(Mx+x) + \sin(Mx-x)] dx$$

$$= \frac{1}{\pi} \int_0^\pi [\sin((M+1)x) + \sin((M-1)x)] dx$$

[⚠ For  $M=1$  THIS BECOMES:

$$B_1 = \frac{1}{\pi} \int_0^\pi \sin(2x) dx = \frac{1}{\pi} \left[ -\frac{\cos(2x)}{2} \right]_0^\pi = \frac{1}{2\pi} [-\cos(2\pi) + \cos(0)] = 0$$

$$= \frac{1}{\pi} \left[ -\frac{\cos((M+1)x)}{M+1} - \frac{\cos((M-1)x)}{M-1} \right]_0^\pi$$

$$= \frac{1}{\pi} \left( -\frac{\cos(\pi(M+1))}{M+1} - \frac{\cos(\pi(M-1))}{M-1} + \frac{\cos(0)}{M+1} + \frac{\cos(0)}{M-1} \right)$$

$$= \frac{1}{\pi} \left( -\frac{(-1)^{M+1}}{M+1} - \frac{(-1)^{M-1}}{M-1} + \frac{1}{M+1} + \frac{1}{M-1} \right)$$

$$= \frac{1}{\pi} \left( \frac{1}{M+1} (-1)^{M+1} + 1 \right) + \frac{1}{M-1} \left( (-1)^{M-1} + 1 \right)$$

$$= \frac{1}{\pi} \left( \frac{1}{M+1} ((-1)^M + 1) + \frac{1}{M-1} ((-1)^M + 1) \right)$$

$$= \frac{1}{\pi} ((-1)^M + 1) \left( \frac{1}{M+1} + \frac{1}{M-1} \right)$$

$$= \frac{1}{\pi} ((-1)^M + 1) \left( \frac{M-1 + M+1}{(M-1)(M+1)} \right)$$

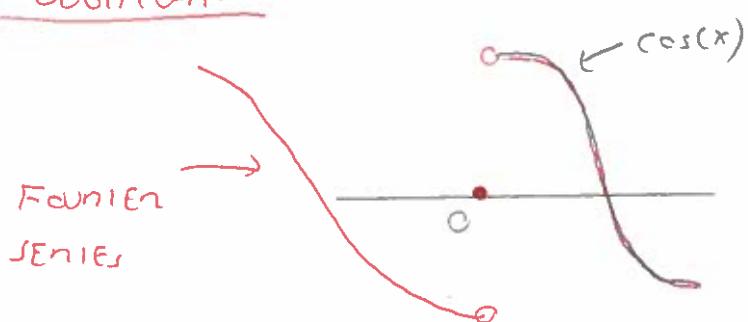
$$B_M = \frac{2}{\pi} ((-1)^M + 1) \left( \frac{M}{M^2 - 1} \right)$$

$$B_M = \begin{cases} \frac{4}{\pi} \frac{M}{M^2 - 1} & \text{IF } M \text{ IS EVEN} \\ 0 & \text{IF } M \text{ IS ODD} \end{cases}$$

(NOTE THAT THIS INCLUDES THE CASE  $M=1$ )

HENCE  $\cos(x) = \sum_{M=1}^{\infty} \frac{2}{\pi} ((-1)^M + 1) \frac{M}{M^2 - 1} \sin(Mx)$

IF WE NAIVELY PLUG IN  $x=0$ , WE GET  $1=0$ , BUT THIS IS NOT A CONTRADICTION, BECAUSE THE ABOVE FOURIER SERIES CONVERGES TO THE ODDIFICATION OF  $\cos(x)$  ON  $(-\pi, \pi)$ .



IN PARTICULAR AT  $x=0$ , THE FOURIER SERIES CONVERGES TO  $0$  SO WE ACTUALLY GET  $0=0$ , WHICH IS NOT A CONTRADICTION!  
YEAH MATH!!!

6. (10 points) Derive Parseval's identity for the following expansion on  $(0, \pi)$

$$1 = \sum_{m=0}^{\infty} A_m \cos\left(\left(\frac{2m+1}{2}\right)x\right)$$

and use it to calculate the sum

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

**Note:** This is a series we haven't seen before. The only thing you're allowed to assume is that the above cosine functions are orthogonal

Since  $\left\{ \cos\left(\left(\frac{2m+1}{2}\right)x\right) \right\}_{m=0}^{\infty}$  is orthogonal,

By the Pythagorean theorem,

$$\|1\|^2 = \left\| \sum_{m=0}^{\infty} A_m \cos\left(\left(\frac{2m+1}{2}\right)x\right) \right\|^2$$

$$= \sum_{m=0}^{\infty} |A_m|^2 \left\| \cos\left(\left(\frac{2m+1}{2}\right)x\right) \right\|^2$$

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Now  $\|1\|^2 = \int_0^{\pi} 1^2 dx = \pi$

And  $\left\| \cos\left(\left(\frac{2m+1}{2}\right)x\right) \right\|^2 = \int_0^{\pi} \cos^2\left(\left(\frac{2m+1}{2}\right)x\right) dx$

$$= \int_0^{\pi} \frac{1}{2} + \frac{\cos\left(2\left(\frac{2m+1}{2}\right)x\right)}{2} dx$$

$$= \left[ \frac{x}{2} + \frac{\sin\left((2m+1)x\right)}{2(2m+1)} \right]_0^{\pi} = \frac{\pi}{2}$$

HENCE WE GET

$$\pi = \sum_{m=0}^{\infty} |A_m|^2 \frac{\pi}{2} \Rightarrow \boxed{\sum_{m=0}^{\infty} |A_m|^2 = 2}$$

$$\text{Now } A_M = \frac{\int_0^{\pi} 1 \cos\left(\left(\frac{2M+1}{2}\right)x\right) dx}{\int_0^{\pi} \cos^2\left(\left(\frac{2M+1}{2}\right)x\right) dx} \Bigg\} \frac{\pi}{2}$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos\left(\left(\frac{2M+1}{2}\right)x\right) dx$$

$$= \frac{2}{\pi} \left[ \sin\left(\left(\frac{2M+1}{2}\right)x\right) \frac{2}{2M+1} \right]_0^{\pi}$$

$$= \frac{4}{\pi(2M+1)} \sin\left(\left(\frac{2M+1}{2}\right)\pi\right)$$

$$= \frac{4}{\pi(2M+1)} \sin\left(\pi M + \frac{\pi}{2}\right) \rightarrow = \begin{aligned} & \sin(\pi M) \cos\left(\frac{\pi}{2}\right) \\ & + \cos(\pi M) \sin\left(\frac{\pi}{2}\right) \\ & = \cos(\pi M) \end{aligned}$$

$$= \frac{4(-1)^M}{\pi(2M+1)}$$

HENCE BY PARSIVAL'S IDENTITY:

$$\sum_{M=0}^{\infty} \left| \frac{4(-1)^M}{\pi(2M+1)} \right|^2 = 2$$

$$\sum_{M=0}^{\infty} \frac{16}{\pi^2} \frac{1}{(2M+1)^2} = 2$$

$$\sum_{M=0}^{\infty} \frac{1}{(2M+1)^2} = \frac{\pi^2}{16} \cdot 2 = \boxed{\frac{\pi^2}{8}}$$

7. (10 points) Show that the only solution of the following PDE for  $0 < x < 1$  is the zero-solution.

$$\begin{cases} u_t = -u_{xxxx} - u^7 \\ u(0, t) = 0, u_x(0, t) = 0 \\ u(1, t) = 0, u_x(1, t) = 0 \\ u(x, 0) = 0 \end{cases}$$

ENERGY METHODMULTIPLY THE PDE BY  $U$  AND INTEGRATE  
WITH RESPECT TO  $X$  FROM 0 TO 1:

$$\int_0^1 U_t U dx = \int_0^1 -U_{xxxx} U dx - \int_0^1 U^7 U dx$$

$$\textcircled{A} = \int_0^1 \frac{1}{2} \frac{d}{dt} (U^2) dx = \frac{d}{dt} \left[ \frac{1}{2} \int_0^1 U^2 dx \right] = E'(t)$$

where  $E(t) = \frac{1}{2} \int_0^1 (U(x, t))^2 dx$

$$\begin{aligned} \textcircled{B} &= \left[ -U_{xxx} U \right]_{x=0}^{x=1} + \int_0^1 U_{xxx} U_x dx \\ &= -U_{xxx}(1, t) U(1, t) + U_{xxx}(0, t) U(0, t) + \int_0^1 U_{xxx} U_x dx \end{aligned}$$

$$\begin{aligned} \text{IBP AGAIN!} &= \left[ U_{xx} U_x \right]_{x=0}^{x=1} - \int_0^1 U_{xx} U_{xx} dx \\ &= U_{xx}(1, t) U_x(1, t) - U_{xx}(0, t) U_x(0, t) - \int_0^1 (U_{xx})^2 dx \\ &= - \int_0^1 (U_{xx})^2 dx \leq 0 \end{aligned}$$

$$\textcircled{C} = - \int_0^1 U^\delta dx \leq 0$$

THE THEREFORE WE GET:

$$\textcircled{A} = \textcircled{B} + \textcircled{C}$$

$$E'(t) = - \int_0^1 (U_{xx})' dx - \int_0^1 U^\delta dx \leq 0$$

$$\Rightarrow E'(t) \leq 0$$

$\Rightarrow E$  IS DECREASING

$$\Rightarrow E(t) \leq E(0) = \frac{1}{2} \int_0^1 (U(x,0))' dx = 0 \\ = 0$$

HENCE  $0 \leq \frac{1}{2} \int_0^1 \underbrace{U'(x,t)}_{\geq 0} dx \leq 0$

$$\Rightarrow \int_0^1 \underbrace{U'(x,t)}_{\geq 0} dx = 0$$

$$\Rightarrow U(x,t) \equiv 0 \quad \text{For all } x \text{ and } t$$

$$\Rightarrow \boxed{U \equiv 0}$$

8. (15 = 7 + 8 points) The grand finale!!!

- (a) **Definition:** If  $\mathbf{F} = (F_1, \dots, F_n)$  is a vector field in  $\mathbb{R}^n$  and  $f$  is a function, then  $f\mathbf{F} = (fF_1, \dots, fF_n)$  (you multiply each component by  $f$ ). Show that

$$\operatorname{div}(f\mathbf{F}) = f(\operatorname{div}(\mathbf{F})) + (\nabla f) \cdot \mathbf{F}$$

SINCE  $f\mathbf{F} = (fF_1, \dots, fF_n)$

$$\operatorname{div}(f\mathbf{F}) = (fF_1)_{x_1} + (fF_2)_{x_2} + \dots + (fF_n)_{x_n}$$

$$= f_{x_1} F_1 + f(F_1)_{x_1} + f_{x_2} F_2 + f(F_2)_{x_2} + \dots + f_{x_n} F_n + f(F_n)_{x_n}$$

$$= f(F_1)_{x_1} + f(F_2)_{x_2} + \dots + f(F_n)_{x_n}$$

$$+ f_{x_1} F_1 + f_{x_2} F_2 + \dots + f_{x_n} F_n$$

$$= f((F_1)_{x_1} + (F_2)_{x_2} + \dots + (F_n)_{x_n})$$

$$+ (f_{x_1}, \dots, f_{x_n}) \cdot (F_1, F_2, \dots, F_n)$$

$$= f \operatorname{div}(\mathbf{F}) + \nabla f \cdot \mathbf{F} \quad \checkmark$$

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- (b) Let  $D$  be a (connected) region in  $\mathbb{R}^n$ . Use (a) to solve the following Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = 0 \text{ in } D \\ \frac{\partial u}{\partial n} = 0 \text{ on bdy } D \end{cases}$$

### ENERGY METHOD

MULTIPLY  $\Delta U = 0$  BY  $U$  AND INTEGRATE ON  $D$ :

$$\int_D \Delta U U \, dx = \int_D 0 \, dx = 0$$

$$\int_D \operatorname{DIV}(\nabla U) U \, dx = 0 \quad (*)$$

NOW APPLY (a) WITH  $f = U$ ,  $F = \nabla U$ :

$$\operatorname{DIV}(U \nabla U) = U \operatorname{DIV}(\nabla U) + \nabla U \cdot \nabla U$$

$$\Rightarrow \operatorname{DIV}(\nabla U) U = \operatorname{DIV}(U \nabla U) - |\nabla U|^2$$

so  $\overset{*}{\underset{D}{\int}} 0 = \int_D \operatorname{DIV}(\nabla U) U \, dx = \int_D \operatorname{DIV}(U \nabla U) \, dx - \int_D |\nabla U|^2 \, dx$

$\operatorname{DIV}$   $\overset{*}{\underset{\text{bdy } D}{\int}} = \int_{\text{bdy } D} U \nabla U \cdot N \, ds - \int_D |\nabla U|^2 \, dx$

THM  $\frac{\partial U}{\partial N}$

$$0 = \int_{\text{BDY } D} U \underbrace{\frac{\partial U}{\partial N}}_{0, \text{ by}} ds - \int_D |\nabla U|^2 dx$$

ASSUMPTION

$$\Rightarrow 0 = - \int_D |\nabla U|^2 dx \Rightarrow \int_D \underbrace{|\nabla U|^2}_{\geq 0} dx = 0$$

$$\Rightarrow |\nabla U|^2 = 0 \text{ EVERYWHERE}$$

$$\Rightarrow \nabla U = 0 \text{ EVERYWHERE}$$

$$\Rightarrow \boxed{U = C} \text{ EVERYWHERE}$$

(so  $U$  is constant)