MATH 112A – FINAL STUDY GUIDE

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General Info

The Final Exam takes place on **Wednesday, December 11, 4-6 pm in 1600 DBH.** Please bring your student ID, as we'll be checking IDs during the exam. There will be a seating chart for the exam, which I'll send out a day or two before the exam.

WARNING: The final will be very very **VERY** hard!!! I'm expecting an average of about 45 %, so please study hard for it and don't underestimate it! That said, remember that your grade will also include your hw/quiz and midterm score, so in the end everything should balance out.
This is the study guide for the exam. Please read it carefully, because it contains a lot of info about what's going to be on the exam and what I expect you to know or not to know. That said, remember that this study guide is just a *guide* and not a complete list. I've tried to make this list as complete as possible, but there are always things that I may have missed. **Some of the expectations have changed compared to the midterm, so it's probably worthwhile to read the whole study guide.**

The final covers sections 1.1-1.6, 2.1-2.4, 3.1-3.2, 4.1-4.2, 5.1, 5.4, 6.1-6.2 inclusive.

Things you absolutely need to know

Here is the **BARE MINIMUM** of what I expect you to know. There will **certainly** be more than that on the exam!

(1) Derive the solution of $au_x + bu_y = 0$
(2) Find the solution of more general first-order PDE like $xu_x+yu_y = 0$
(3) The Coordinate Method (**I will NOT give you the coordinates any more,** with some exceptions)
(4) Find the general solution of $u_{tt} = c^2u_{xx}$, using the Factoring Method and the Coordinate Method.
(5) Derive d'Alembert's Formula

Date: Wednesday, December 11, 2019.

(6) Do (4) and (5) with more general PDE like and $u_{xx} - 3u_{xt} - 4u_{tt} = 0$
(7) You do **NOT** need to memorize the formula for $S(x,t)$, but you need to know how to derive it, starting from the point: Suppose $u(x,t) = \frac{1}{\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right)$ and plug this into $u_t = k u_{xx}$. You don't need to find the constant C.
(8) Energy methods, both for the wave and the heat equation, and deriving uniqueness from that. **I will NOT tell you which function to multiply your PDE by any more!**
(9) Maximum principle (for the heat equation **and** Laplace's equation), and deriving uniqueness from that.
(10) Solve PDEs by changing functions, things like Let $v = e^{-u}$, what PDE does v solve? See 2.4.16 for example.
(11) Derive the solutions for the heat and wave equations on the half-line
(12) Solve the heat, wave, and Laplace equations with separation of variables
(13) Derive the Fourier sine and cosine coefficients of a function, including complex Fourier series
(14) Derive Parseval's identity
(15) Show Laplace's equation is invariant under rotation (in 2 dimensions)
(16) Find the Laplacian in polar coordinates and derive the fundamental solution in 2 dimensions from it

Note: 1.1.11 means Problem 11 in section 1.1.

SECTION 1.1: WHAT IS A PARTIAL DIFFERENTIAL EQUATION

- Solve some simple PDE like $u_x = 0$, $u_{xx} = 0$, $u_{xy} = 0$
- Determine if if a PDE is linear or not (1.1.2, 1.1.3)
- Find the order of a PDE, and determine if it's homogeneous or not
- Know the fact that the general solution of $L(u) = f$ is equal to the general solution of $L(u) = 0$ + a particular solution of $L(u) = f$
- Verify if a function solves a PDE (1.1.11, 1.1.12)

SECTION 1.2: FIRST-ORDER LINEAR EQUATIONS

- Find the solution of $au_x + bu_y = 0$ using the geometric method
- In particular, know how to solve the transport equation $u_t + cu_x = 0$
- Do the same thing with the coordinate method. **This time, I will not give you the coordinates, you should have enough practice by now to guess them.** Also try 1.2.13, but do that problem without the inhomogeneous term

- Of course, for this you need to be super comfortable with the Chain Rule/Chen Lu
- Also know how to find the solution of some more general first-order PDEs, like $u_x + yu_y = 0$ or $u_x + 2xy^2u_y = 0$ and others (1.2.5)
- Those problems require you to be comfortable with differential equation techniques like solving $\frac{dy}{dx} = \frac{y}{x}$!
- For all those problems, I might ask you to plug in initial conditions, like 1.2.1
- Solve PDEs by substituting new functions, as in 1.2.2 and 1.2.8 (in that case I would tell you what function to use, like the HW 2 hint)
- Also look at 1.2.9

SECTION 1.3: FLOWS, VIBRATIONS, AND DIFFUSIONS

- **This time KNOW** the derivation of the transport equation $u_t + cu_x = 0$ (see Lecture 4 or Example 1 in 1.3), and know how to find its solutions
- Skip the rest of the section

SECTION 1.4: INITIAL AND BOUNDARY CONDITIONS

- Know the different kinds of boundary conditions (Dirichlet, Neumann, Robin), both in 1 dimension and in higher dimensions
- Know what a normal vector is, and know the normal derivative $\frac{\partial u}{\partial n}$.
- Know the definition of $div(F)$, where F is a vector field, and know the divergence theorem $\int_D F \cdot ndS = \int_D div(F)dx$
- You don't need to know all those interpretations of the boundary conditions that the book gives you in 1.4

SECTION 1.5: WELL-POSED PROBLEMS

- Know what existence, uniqueness, and stability mean. Although you don't need to read the discussion in the book, do check out the homework problems (like 1.5.4 - 1.5.6), and the material I covered in Lecture 5

SECTION 1.6: TYPES OF SECOND-ORDER EQUATIONS

- Know the definition of elliptic, hyperbolic, parabolic and figure out if a PDE is elliptic etc. (1.6.1, 1.6.2, 1.6.4, 1.6.6)
- Note that the book and I use slightly different conventions, so decide which one you like more

SECTION 2.1: THE WAVE EQUATION

- Absolutely know how to derive the general solution of the wave equation $u_{tt} = c^2 u_{xx}$ using the factoring method.
  Note: At some point you have to solve $u_t + cu_x = f(x + ct)$. In that case, for the particular solution, guess $u(x,t) = aF(x+ct)$ and solve for a
- Do the same thing but with the coordinate method. **Again, this time I won't tell you which coordinates to use.**
- Know how to derive D'Alembert's formula. Again, the book and I give slightly different derivations, so decide which one you like more (2.1.1, 2.1.2)
- You also need to know how to use the factoring method to solve more general PDE like $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ (2.1.9, 2.1.10) **Note:** Just be careful (for instance) that an antiderivative of $(F(4s))'$ with respect to s is $\frac{1}{4}F(4s)$ and not $F(4s)$.
- Ignore the plucked string example (Example 2 in 2.1), it's really just for illustrative purposes
- Also check out 2.1.8, it's fun!

SECTION 2.2: CAUSALITY AND ENERGY

- Ignore domain of dependence/domain of influence, I won't ask anything about it!
- Know how to use energy methods. This means multiplying your PDE by a function and integrating by parts with respect to x.
  Note: I will **NOT** give you the energy here, and you should **NOT** memorize it, but you DO have to know how to derive it. **I will not tell you which function to multiply the PDE any more!**
- Don't just expect the wave equation, I might give you a different PDE for which the energy method holds, like the problem on the midterm.
- Use the energy method to prove that the only solution of the wave equation with zero initial conditions is the zero function (2.2.1) and know how to derive uniqueness of solutions of the wave equation from that (see Lecture 8)
- Check out 2.2.2 and 2.2.5

SECTION 2.3: THE DIFFUSION EQUATION

- Know the statement of the Maximum Principle, but you don't need to know its proof
- Use the maximum principle to show that the only solution of the heat equation with 0 initial and boundary conditions is the zero

function (Lecture 12) and know how to derive uniqueness from that (Lecture 12), as well as stability in the max-sense (Lecture 12)

- Prove the comparison principle for the heat equation (2.3.6)
- Also check out 2.3.4, it's a lot of fun :)
- Know how to use energy methods for the heat equation (and possible variations of the heat equation). **I will not tell you which function to multiply your equation with!** (check out 2.4.15 as well), and use this to derive uniqueness (Lecture 11) and stability in the integral sense (Lecture 11)

SECTION 2.4: DIFFUSION ON THE WHOLE LINE

- You need to know how to derive the fundamental solution $S(x,t)$ in the following sense: Suppose $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{\sqrt{t}}\right)$ for some $v = v(y)$ and α to be found, plug this into $u_t = ku_{xx}$, find α and solve for v. I highly recommend looking at Lecture 9 instead of the book, since the book's derivation makes no sense
- You do **NOT** need to memorize the formula for $S(x,t)$, it will be provided to you if necessary
- Know the definition of convolution and be able to write a solution of $u_t = ku_{xx}$ with $u(x,0) = \phi(x)$ in terms of S and ϕ
- Know how to show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (see Gaussian Integral)and that $\int_{-\infty}^{\infty} S(x,t) dx = 1$ (given the formula for S) (2.4.6, 2.4.7)
- Don't memorize the formula of Erf (in the book), it will be given to you if necessary
- I might ask you to find an explicit formula of u if ϕ is given, as in Example 2 in the book or as in Lecture 10. The whole point is just to complete the square with respect to y (2.4.3)
- You don't need to understand the part of Lecture 10 with the Dirac Delta functional, and you don't need to understand the intuition behind convolution (Lecture 10)
- Also check out 2.4.11(a)(b) and 2.4.16-18, those are all good problems.
- Know the lyrics to the song I sang in class ☺

Note: Although don't need to know section 2.5 (Comparison of Waves and Diffusions), you should know what the terms mean (like speed of propagation, maximum principle, well-posed), and Lecture 13 is a great review of all the properties. It might also be useful to look at 2.5.1 and 2.5.3.

Section 3.1: Diffusion on the half-line

- Know the definition of even, odd, evenification, oddification
- Derive the formula of solutions of the heat equation on the half-line, both for Dirichlet (Lecture 16) and Neumann boundary conditions (3.1.3). Do **NOT** memorize that formula, but know how to apply it in examples (see Example 1 in 3.1, as well as 3.1.1 and 3.1.2).
- I might ask to write everything in terms of Erf; in that case would give you the definition of Error function.

Section 3.2: Reflection of waves

- Derive the formula of solutions of the wave equation on the half-line, both for Dirichlet (Lecture 17) and Neumann boundary conditions (3.2.1). Again, do **NOT** memorize that formula, but know how to apply it in examples (like 3.2.5)
- Ignore the section on 'The finite interval'
- For both the heat and wave equations, know how to solve inhomogeneous problems (see material at the end of Lecture 17, or AP1 and AP4 in HW 5)
- Ignore AP2 in HW 5, but know AP3 (it's useful for Fourier series)

Section 4.1: Separation of Variables

- Know how to apply separation of variables to find a solution to a PDE, like the heat, wave equation, or Laplace equation.
- Do **NOT** skip the 3 cases!!! I will be **VERY** picky on the exam about the way you write out the solution of your problem, so be as thorough as you can be. This is your one and only warning!
- **WARNING: I might give you a completely different PDE, so don't just expect me to ask a wave equation problem for example!**
- Again, beware that the book and I have different conventions (the book uses $-\lambda$ where I use λ). It doesn't matter because in the end our solutions are the same
- Problems 4.1.2, 4.1.3, 4.1.4, 4.1.6 are all fair game for the exam, as well as AP1 and AP2 in HW 6. For AP2, I would tell you to guess $u(x,t) = X(x) + T(t)$

Section 4.2: The Neumann Condition

- Same instructions and warnings as 4.1, all problems in that section are fair game for the exam

- Also know how to solve problems with inhomogeneous boundary conditions, see the end of Lecture 20, as well as AP1 in HW 6.

SECTION 5.1: THE COEFFICIENTS

- Know the definition of orthogonal vectors and know the 'hugging' formula (see Lecture 21)
- Show that $\{\sin(mx)\}$ is orthogonal on $(0, \pi)$ and similarly with $\{\cos(mx)\}$ and even with $\{\cos(mx), \sin(mx)\}$ (see for instance AP3(a) in HW 7) You do **NOT** need to memorize the identities for $\sin(A) sin(B)$, I would give them to you.
- Show that $\int_0^\pi \sin^2(mx)dx = \frac{\pi}{2}$ (see AP3(b) in HW7, but I wouldn't give you the hint)
- Derive the formulas for the Fourier sine series, the Fourier cosine series, and the Full Fourier series. Beware of the case $m = 0$ and beware that the Full series has a slightly different formula. You should memorize them, but remember the hugging analogy. Don't forget about the factor of 2.
- Also check out AP1 in HW 7, which gives an alternate derivation
- Calculate the Fourier sine (and cosine and full) coefficients of a function, like all the examples in 5.1, and 5.1.2
- **Note:** The D/I method (aka tabular integration) helps to find coefficients when f is a polynomial, see Lecture 21
- Use the Fourier coefficients to solve the heat/wave/Laplace equation, like Example 6 in 5.1 or AP2 in HW7
- You also need to know complex Fourier series (Lecture 22): Know how to derive the coefficients and know how to do examples with them, see also 5.2.11.

SECTION 5.4: COMPLETENESS

- This section has lots of useless information that you don't need to know. Do not read 5.4 in the book, but instead look at my Lecture 23 notes.
- The only two things you need to know is the fact that the Fourier series converges to f whenever f is continuous (and converges to the average of the jump of f if it has one), and Parseval's identity
- Given f, draw the graph of the Fourier sine/cosine/full series of f. You might need to evenify or oddify f. See the example given in Lecture 23, as well as 5.4.5(b)(d) (For (d), I'd tell you which point to plug in, but you'd have to justify the convergence) and 5.4.6
- Know the Pythagorean Theorem for orthogonal vectors

- Know how to derive Parseval's identity by using the (infinite) Pythagorean Theorem. Do **NOT** memorize the formula; either you'll have to rederive it, or I'd give it to you.
- Use Parseval's identity to derive fun sums, like the sum of $\frac{1}{n^2}$ or the sum of $\frac{1}{n^4}$. In that case, I would give you the function and the interval, but you'd have to guess whether to use sine or cosine (or full) series.
- Also check out AP4 in HW7 and 5.4.14

Section 6.1: Laplace's Equation

- In 2 dimensions, show that Laplace's Equation is invariant under rotations (Lecture 25 or page 156 in 6.1) I wouldn't give you the formulas for x' and y', you'd have to know them
- You do **NOT** need to know how to show rotation-invariance in higher dimensions (although it's a fun linear algebra exercise ☺)
- Know how to derive Laplace's equation in polar coordinates. You can either use my method or the book's method. Do **NOT** memorize the formula!
- Derive the Fundamental Solution of Laplace's Equation in 2 dimensions. In that case, I would give you the formula for the Laplacian in polar coordinates (or ask you to rederive it). You do not need to know how to choose the value of the constant C
- You do **NOT** need to know or derive the Laplacian in spherical coordinates, but you need to know how to derive the fundamental solution **given** the Laplacian in spherical coordinates (see start of Lecture 27). Again, you don't need to know how to find the constant
- Know how to solve $-\Delta u = f$ given the fundamental solution (see Lecture 27; it's just $S \star f$).
- I could ask you to derive the fundamental solution in n dimensions using the trick in AP1 in HW 8
- You don't need to know the definition of subharmonic, but know how to do AP2 in HW8
- Also check out 6.1.2, 6.1.5, 6.1.7, those are good practice problems with the polar/spherical Laplace
- Know the statement of the mean-value formula (see Lecture 26)
- Know the statement of the maximum principle (both weak and strong versions), and know how to prove it using the mean-value formula (see Lecture 26). You **DON'T** need to know how to prove it the book's way. Also check out 6.3.1, which is related to this
- Know how to derive some consequences of the maximum principle, such as uniqueness and positivity (see Lecture 26)

- You **DON'T** need to know the Applications of Laplace's equation I presented in Lecture 27, and not even the OMG Application in Lecture 28; they are mainly for fun
- Again, know the definition of $div(F)$, as well as the divergence theorem $\int_D div(F)dx = \int_{bdyD} F \cdot ndS$, and know the fact that $\Delta u = div(\nabla u)$, also check out 6.1.11 for an application of the divergence theorem
- Know how to derive Laplace's equation in the following sense: I would tell you that $\int_{bdyD} F \cdot ndS = 0$ for every D, as well as $F = -c\nabla u$

SECTION 6.2: RECTANGLES AND CUBES

- Solve Laplace's equation $u_{xx}+u_{yy} = 0$ in a rectangle. Remember to use $\cosh$ and $\sinh$ this time, instead of exponential functions. Also beware that sometimes you have to do the 3 cases on Y instead of X (always choose the one with 0 boundary conditions). Check out 6.1.3 and 6.1.7a for practice
- If none of the boundary conditions are zero, then you have to split it up into 2 equations, like the end of Lecture 24 or 6.1.1 an 6.1.4 for instance
- You do **NOT** need to know how to solve Laplace's equation on a cube, **BUT** know how to derive the coefficients of a double Fourier series (like equation 8 on page 164)