MATH 3A – FINAL STUDY GUIDE

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GENERAL INFO

The Final Exam takes place on **Friday, December 13, 1:30 - 3:30 pm in 178 Humanities Hall.** Bring your student ID, as we'll be checking IDs during the exam.

This is the study guide for the exam. Please read it carefully, because it contains a lot of info about what's going to be on the exam and what I expect you to know or not to know. That said, remember that this study guide is just a *guide* and not a complete list. I've tried to make this list as complete as possible, but there are always things that I may have missed.

The final exam covers 1.1-1.9 (ignore 1.6), 2.1-2.3, 2.8-2.9, 3.1-3.3, 5.1-5.4, 6.1-6.6

WARNING: The final will be **HARDER** than the midterm, so please don't underestimate it. I'm expecting roughly a 60 % average (similar to the Winter 2019 finals), but there will be some trickier questions. Moreover, the final is cumulative, meaning you also have to look at the practice **midterms**, and the actual midterm you took. Roughly 70 % of the exam will have questions from chapters 5 and 6, and 30 % chapters 1-3. As before, the practice exams on my website are a bit easier than the exam will be, so don't *just* study the practice exams. Also look at this study guide and the suggested problems. If you want some challenging problems, the old quizzes on my website have some trickier problems!

The exam will have a mix of computational questions, True/False questions (some where you don't have to justify your answer and others where you do), some small proofy questions (think like the ones on the homework or on the True/False) And there will be some *very* interesting questions too ☺

Beware: In lecture, I tried to hit the main points of each section, but there are topics in the book and/or the suggested homework that I didn't cover in

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lecture but that I expect you to know for the exam. That's why it's especially important that you look at this study guide, in order to avoid surprises. The *best* way to prepare for the exam is to (1) review your notes from lecture, (2) look at the suggested homework, (3) look at the YouTube videos I posted (except the ones that are labeled optional), (4) read the sections in the book, and (5) do the practice exams on my website. You don't need to study the quizzes or the discussion worksheets, since I didn't look at them at all.

Note: 1.1.11 means Problem 11 in section 1.1. Remember that there is a hint/solution-bank on my website as well.

YouTube Playlists

There are a lot of videos on my YouTube channel, based on the concepts covered in this course. Check them out if you need help with a topic:

- Chapter 1: Linear Equations in Linear Algebra
- Chapter 2: Matrix Algebra
- Chapter 3: Determinants
- Chapter 5: Eigenvalues and Eigenvectors
- Chapter 6: Orthogonality and Least-Squares
- 111 Linear Algebra True/False Question

Chapter 1: Linear Equations in Linear Algebra

- Solve a system of equations, or determine if there are no solutions. Try to write your solutions in vector form. (1.1.11, 1.1.13, 1.2.11, 1.2.13, 1.4.11, 1.5.9, 1.5.12, Gaussian Elimination, No solutions, Infinitely many Solutions)
- Find values of h for which a system has a solution (1.1.21, 1.1.28)
- Know the existence/uniqueness theorem (Theorem 2 in section 1.2), it's a great way of checking if a system has a solution or not. See in particular 1.2.24 and 1.2.25.
- Determine if a given vector $\mathbf{b}$ is a linear combination of other vectors (1.3.11, 1.3.13, 1.3.15, 1.3.17, Span)
- In section 1.3, ignore the part on 'Linear Combinations in Applications'
- Also look at 1.3.25 and 1.3.26, people find that tricky.
- Know Theorem 6 in section 1.5; it's a very useful way of thinking of solutions of systems; in theory know how to prove this, see 1.5.25.
- Determine whether a set of vectors is linearly dependent or independent (1.7.5, 1.7.7, 1.7.11, 1.7.15, 1.7.17, Linear Independence)
- You don't need to know Theorem 7 in 1.7 if it's too confusing to you, but know Theorems 8 and 9, they're useful.

- Show that T is or is not a linear transformation (1.8.32, 1.8.32, Linear Transformations).
- I could ask you to show that, for any matrix A, $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.
- Ignore example 6 in 1.8
- Given a linear transformation T and a vector $\mathbf{b}$, determine whether $\mathbf{b}$ is in the image of T (1.8.3, 1.8.9)
- Know how to derive the formula for the rotation matrix (Example 3 in 1.9), but you need to memorize it (check out Rotation Matrix if you want)
- Do **NOT** memorize Tables 1-4 in 1.9
- Find the matrix of a given linear transformation T (1.9.1, 1.9.3, 1.9.5, 1.9.9, 1.9.11, 1.9.17)
- Determine if a linear transformation is one-to-one or onto (1.9.25, One-to-one, onto, matrix)
- Know the Row Theorem from lecture, and remember that rows, $A\mathbf{x} = \mathbf{b}$, span, and onto go together; check out Row Theorem and ax + by = e
- Know the Column Theorem from lecture, and remember that columns, $A\mathbf{x} = \mathbf{0}$, linear independence, and one-to-one go together.

Chapter 2: Matrix Algebra

- Given A and B, find AB, or say 'it does not exist' (2.1.5, 2.1.6, 2.1.9), Matrix Multiplication, AB vs. BA
- Calculate things like A^T, A^2 etc.
- Understand what AB means in terms of linear transformations
- Mnemonic: **In**put, **M**outhput, so $\mathbb{R}^n$ to $\mathbb{R}^m$
- You don't need to understand example 6 in section 2.1
- Remember that $(AB)^T = B^T A^T$ (reverse order)
- Find the inverse of a matrix A, including the formula in the 2×2 case, and the general procedure where you form the big matrix $[A|I]$ and row-reduce (2.2.1, 2.2.3, 2.2.31, 2.3.3, 2.3.7, Calculating A^{-1})
- Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ (2.2.5)
- Remember that $(AB)^{-1} = B^{-1}A^{-1}$ (reverse order)
- Understand what A^{-1} means in terms of linear transformations.
- Ignore example 3 in 2.2
- Know the three types of elementary matrices and their inverses, but you **don't** need to know how to write a matrix in terms of elementary matrices, and you don't need to know the proof of Theorem 7.
- Ignore the section on 'Another View of Matrix Inversion'

- Understand all the implications of the invertible matrix theorem (IMT). You don't need to memorize them (I won't ask you 'List all the conditions of the IMT'), but know how to recognize it. For example, I could ask you "If A is $n \times n$ and the columns of A span $\mathbb{R}^n \cdots$," then in your mind you should be like "Ah! The IMT!" (2.3.15−29)
- Figure out when a matrix or a linear transformation is invertible (2.3.5, 2.3.33, A invertible)
- Always remember that the IMT says 'An invertible matrix is awesome,' so what you want to be true for an invertible matrix is true for an invertible matrix
- **Unless otherwise specified, do NOT assume A is square!!!** This is a main source of pitfalls!
- You don't need to know how to prove the IMT.
- Show H is or is not a subspace of $\mathbb{R}^n$ (2.8.1, 2.8.2, Subspace, Not a subspace)
- I could ask you to show that $Span\{\mathbf{u}, \mathbf{v}\}$ is a subspace, see Span is a subspace
- I could ask you to show that $Nul(A)$ is a subspace of $\mathbb{R}^n$ (Theorem 12 in 2.8), see Nul(A) is a subspace or $Col(A)$ is a subspace of $\mathbb{R}^m$ (simply because it's a span)
- Given A, find a basis for $Nul(A)$, find a basis for $Col(A)$, find $rank(A)$ and find $dim(Nul(A))$ (2.8.25, 2.8.26, 2.9.9, 2.9.12, Nul(A), Nul(A), Col(A), Rank(A)).
- Remember that for $Col(A)$, you need to go back to the columns of A, and for $Nul(A)$, you need the RREF.
- You can ignore any problems with $Row(A)$.
- Check if something is a basis or not (2.8.17, 2.8.20, Basis check)
- Given $\mathbf{x}$, find $[\mathbf{x}]_{\mathcal{B}}$ and vice-versa (2.9.1, 2.9.3, Coordinates)
- Find a basis and the dimension of the span of vectors (2.9.13, Basis and dimension) (it all boils down to finding a basis for the column space)
- Do problems involving the rank-nullity theorem (2.8.19-25)
- Intuitively, $Nul(A)$ measures how bad a matrix is, and $Col(A)$ or $rank(A)$ measures how good a matrix is. The rank-nullity theorem says that their sum balances out, like a conservation of energy.
- Don't worry about the Basis Theorem (Theorem 15 in 2.9, also known as the Dimension Theorem in lecture), although it's kind of useful
- For the invertible matrix theorem (section 2.9), again you don't *have* to memorize it, but understand what it's saying

CHAPTER 3: DETERMINANTS

- Calculate the determinant of a matrix, possibly using row-reductions (3.1.31, 3.1.3, 3.1.9, 3.1.13, 3.2.5, 3.2.7, 3.2.13, Determinants, Determinants and row-reduction, Another determinant)
- Use determinants to find if a matrix is invertible (3.2.23, 3.2.24, Determinants and Invertibility)
- In section 3.2, ignore the paragraph preceding Theorem 4, as well as the sections on 'A Linearity Property of the Determinant Function' and 'Proofs of Theorems 3 and 6'
- Solve questions using the fact that $\det(AB) = \det(A)\det(B)$ and $\det(A^T) = \det(A)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ (3.2.31−35)
- Solve a system using Cramer's rule (3.3.1, 3.3.3, 3.3.5, Cramer's Rule)
- Ignore the section 'Application to Engineering'
- Find inverses using determinants (3.2.13, 3.2.18. A-1 using determinants)
- In the sections on 'Determinants as Area or Volume' and 'Linear Transformations,' ignore the lengthy descriptions. You just need to know how to find areas of parallelograms (Example 4) and parallelipipeds, as well as the formula in Theorem 10 and Example 5 (3.3.19, 3.3.23, 3.3.27)
- Calculate volumes using determinants (3.3.29, 3.3.20, 3.3.31, 3.3.32, Determinants and Volumes)
- I loooooooooove determinants :)

CHAPTER 5: EIGENVALUES AND EIGENVECTORS

- Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$, or say A is not diagonalizable (5.2.9, 5.2.11, 5.3.9, 5.3.11, 5.3.13, 2×2 example, 3×3 example)
- **Beware:** The book gives you the eigenvalues, but I might ask you to find it, like the problem on the final exam review session.
- Show that a given matrix is not diagonalizable (5.3.8, 5.3.11, Not Diagonalizable)
- Remember the example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it's a great example of a non-diagonalizable matrix.
- Remember that A is diagonalizable if and only if A has n (linearly independent) eigenvectors (5.3.23, 5.3.24)
- Remember that if A has n different eigenvalues, then it is diagonalizable.

- Know that a matrix is invertible if and only if 0 is not an eigenvalue of it.
- Know how to show that nonzero eigenvectors corresponding to different eigenvalues are linearly independent (Theorem 2 in 5.1, although I would only ask you for the case of 2 eigenvectors)
- Here's the Legend of Zelda analogy for diagonalization: Legend of Zelda
- Use the decomposition $A = PDP^{-1}$ to find A^k for any k (5.3.1, 5.3.3)
- Understand the applications of diagonalization presented in lecture (Pokemon Battle, Fibonacci Numbers, Matrix Limit). Each of those applications is fair game for the exam. You don't need to know how to set up the matrices in the second and third examples; they would be given to you. Notice that what the 3 examples have in common is that you find A and calculate A^n.
- You might also want to check out the following videos, they are great practice with the techniques from the 3 applications: Square root of A, e^A, $\cos(A)$, Another Matrix Limit
- Find the $\mathcal{B}-$matrix of a linear transformation and understand why $A = PBP^{-1}$. Also find a basis such that your $B-$matrix is diagonal (5.4.11 -17, $\mathcal{B}-$matrix of A)
- Know the definition of similar matrices and how to do problems with them (5.4.19 $-$ 22)
- No complex eigenvalues (5.5) on the exam

CHAPTER 6: ORTHOGONALITY AND LEAST SQUARES

- Calculate dot products, lengths, distances of vectors (6.1.1, 6.1.10, 6.1.13)
- Know how to derive the 4 fun inequalities from lecture: The Pythagorean Theorem (memorize and derive), the Parallelogram Identity (just derive, see 6.1.24), the Triangle Inequality (memorize and derive with hints), and the Cauchy-Schwarz identity (just derive with hints, or see Cauchy-Schwarz)
- Determine if a set is orthogonal, or orthonormal (6.2.3, 6.2.17, 6.2.21, Orthogonal Sets)
- Find vectors which are orthogonal to a set of vectors (see Theorem 3 in 6.2). You don't need to know what a row-space is. Equivalently: given a basis for W find a basis for $W^{\perp}$. Also know the facts that $(Col(A))^{\perp} = Nul(A^T)$ and $(Nul(A))^{\perp} = Col(A^T)$ (Just think 'put that frown upside-down)
- Express $\mathbf{x}$ as a linear combination of orthogonal vectors (6.2.7, 6.2.9)

- Know how to show that an orthogonal set without $\mathbf{0}$ is linearly independent.
- Remember the hugging formula! It appears both in the question above, and in the question about orthogonal projection.
- Find the orthogonal projection of $\mathbf{x}$ on a subspace W. Use this to write $\mathbf{x}$ as a sum of two orthogonal vectors, and to find the smallest distance between $\mathbf{x}$ and W (6.2.11, 6.3.3, 6.3.5, 6.3.1, 6.3.7, 6.2.15, 6.3.11, Orthogonal Projections)
- I might ask you to prove 6.2.25
- Know the difference between Q^TQ (the identity) and QQ^T (projection on $Col(Q)$) if the columns of Q are orthonormal
- Know the definition of an orthogonal matrix. Remember that orthogonal matrices are **square**
- Use the Gram-Schmidt process to produce an orthogonal or orthonormal basis of a subspace W spanned by some vectors (6.4.3, 6.4.9, 6.4.11, Gram-Schmidt)
- Find the QR decomposition of a matrix (6.4.15, QR-decomposition)
- Find the least-squares solution (and least-squares error) of an inconsistent system of equation. You need to know both methods: by calculating orthogonal projections (remember that the columns of A must be orthogonal to do this) and by multiplying by A^T (6.5.3, 6.5.7, 6.5.9, 6.5.11, Least-Squares)
- Use $A = QR$ to find the least-squares solution of a system (6.5.15)
- You might also want to check out 6.5.19 if you haven't done so already
- Find the equation of a line that best fits points in a least-squares sense (6.6.1, 6.6.3, 6.6.4, Linear Models)

Note: If you're done with studying and need more practice (this is completely optional), you can look at some of the problems in section 7.1, like 7.1.13, 7.1.17, 7.1.19, or see Symmetric Matrices. It looks like a new concept, but it's just putting diagonalization and Gram-Schmidt together.

TRUE/FALSE EXTRAVAGANZA

Do the following set of T/F questions: 1.1.23, 1.1.24, 1.2.21, 1.2.22, 1.3.23, 1.3.24, 1.4.23, 1.4.24, 1.5.23, 1.5.24*, 1.7.21*, 1.7.22, 1.8.21*, 1.8.22, 1.9.23, 1.9.24*, 2.1.15, 2.1.16, 2.2.9, 2.2.10, 2.3.11, 2.3.12, 2.8.21*, 2.8.22, 2.9.17, 2.9.18, 3.1.39, 3.1.40, 3.2.27*, 3.2.28, 5.1.21*, 5.1.22, 5.2.21*, 5.3.21*, 5.3.22, 6.1.19*, 6.1.20, 6.2.23*, 6.2.24, 6.3.21*, 6.3.22, 6.4.17, 6.4.18, 6.5.17, 6.5.18 (The ones with * next to them have solutions in the Homework-Hints)

CONCEPTS

Understand the following concepts. In theory, also know the definitions of the concepts with *

- Pivots
- Row-echelon and reduced row-echelon form
- Span*
- Linear independence*
- Linear Transformation*
- One-to-one* and onto*
- Invertible matrix*
- Invertible Matrix Theorem
- Subspace*
- $Nul(A)$*, $Col(A)$*
- Dimension*
- Rank*
- Rank Theorem*
- Coordinates of $\mathbf{x}$ with respect to $\mathcal{B}$*
- Determinant
- Eigenvalues*, Eigenvectors*, Characteristic polynomial (5.1 - 5.3)
- Diagonalizable*, Diagonalization Theorem (Theorem 5 in section 5.3)
- B Matrix of A (5.4)
- A is similar to B* (5.4)
- Dot product, Length, Orthogonal*, Orthonormal*, Orthogonal Matrix* (6.1)
- $W^{\perp}$* (6.1)
- Orthogonal projection (6.2, 6.3)
- QR decomposition* (6.4)
- Least-squares solution, Least-squares error (6.5)