Math 54. Solutions to Sample Final Exam

1. (12 points) Find the inverse of the matrix $A = \begin{bmatrix} 7 & 10 & -9 \\ 1 & 2 & -3 \\ -1 & 1 & -6 \end{bmatrix}$, if it exists. Use the algorithm from the book (or from class).

The algorithm consists of row-reducing the matrix $[\,A \quad I\,]$:

$$\begin{bmatrix} 7 & 10 & -9 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 1 & 0 \\ -1 & 1 & -6 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 7 & 10 & -9 & 1 & 0 & 0 \\ -1 & 1 & -6 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -4 & 12 & 1 & -7 & 0 \\ 0 & 3 & -9 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3 & -\frac{1}{4} & \frac{7}{4} & 0 \\ 0 & 3 & -9 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3 & -\frac{1}{4} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & -\frac{17}{4} & 1 \end{bmatrix}$$

The matrix is singular, because the first three entries of the bottom row are zero.

2. (20 points) Let A be the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$, where x is a real number.

(a). For which values of x is A similar to a (real) diagonal matrix? (Do not diagonalize the matrix.)

The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ x & -\lambda \end{vmatrix} = \lambda^2 - x\,.$$

If $x > 0$ then A has two distinct real eigenvalues $\pm\sqrt{x}$, so it is diagonalizable. If $x < 0$ then A has (non-real) complex eigenvalues $\pm i\sqrt{-x}$, so it is not diagonalizable. If $x = 0$ then A has a double eigenvalue $\lambda = 0$, for which the eigenspace is

$$\operatorname{Nul} A = \operatorname{Nul} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}\,.$$

This matrix has only one linearly independent eigenvector, so it is not diagonalizable.

Therefore, A is similar to a (real) diagonal matrix if and only if $x > 0$.

(b). For which values of x is A orthogonally diagonalizable?

The real matrix A is orthogonally diagonalizable if and only if it is symmetric, so it is orthogonally diagonalizable if and only if $x = 1$.

3. (20 points) Each of the following parts gives vector spaces V and W, bases $\mathcal{B}$ for V and $\mathcal{C}$ for W, and a linear transformation $T\colon V \to W$. In each case find the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$.

(a). $V = W = \mathbb{R}^2$, $\mathcal{B} = \{(1,1),(-1,1)\}$, $\mathcal{C} = \{(1,0),(0,1)\}$, and T is counterclockwise rotation by 90 degrees.

For solutions, write vectors in horizontal notation, and write $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.
We have

$$M = \begin{bmatrix} T(\vec{b}_1)_{\mathcal{C}} & T(\vec{b}_2)_{\mathcal{C}} \end{bmatrix} = [\,(-1,1)_{\mathcal{C}} \quad (-1,-1)_{\mathcal{C}}\,] = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

(b). $V = W = \operatorname{Span}\{\sin x, \cos x\} \subseteq C[0,2\pi]$, $\mathcal{B} = \mathcal{C} = \{\sin x, \cos x\}$, and T is the linear transformation taking a function to its derivative.

$$M = [\,T(\sin x)_{\mathcal{C}} \quad T(\cos x)_{\mathcal{C}}\,] = [\,(\cos x)_{\mathcal{C}} \quad (-\sin x)_{\mathcal{C}}\,] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c). $V = \mathbb{R}^3$, $W = \operatorname{Span}\{(1,2,3)\} \subseteq \mathbb{R}^3$, $\mathcal{B}$ is the standard basis of $\mathbb{R}^3$, $\mathcal{C} = \{(1,2,3)\}$, and T is the projection of a vector in $\mathbb{R}^3$ to the line through $(1,2,3)$.

Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \operatorname{proj}_{\vec{v}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{14}\vec{v} = \begin{bmatrix} 1/14 \\ 2/14 \\ 3/14 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \operatorname{proj}_{\vec{v}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{2}{14}\vec{v} = \begin{bmatrix} 1/7 \\ 2/7 \\ 3/7 \end{bmatrix},$$

and

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \operatorname{proj}_{\vec{v}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{14}\vec{v} = \begin{bmatrix} 3/14 \\ 6/14 \\ 9/14 \end{bmatrix}.$$

Therefore

$$M = \left[\, T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathcal{C}} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)_{\mathcal{C}} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)_{\mathcal{C}} \,\right] = [\,\tfrac{1}{14} \quad \tfrac{2}{14} \quad \tfrac{3}{14}\,].$$

4. (25 points) Let A be the 4×3 matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 5 & 5 \\ 0 & 2 & 1 \\ 2 & -4 & 0 \end{bmatrix}$.

Write $A = QR$, where Q is a matrix whose columns form an orthonormal basis for $\operatorname{Col} A$ and R is an upper triangular invertible matrix with positive entries on its main diagonal.

First use the Gram-Schmidt process to find Q:

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{bmatrix} 0 \\ 5 \\ 2 \\ -4 \end{bmatrix} - \frac{-18}{9}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} - \frac{-9}{9}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \frac{9}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

After normalizing each of these vectors, we have

$$Q = \begin{bmatrix} 1/3 & 2/3 & 0 \\ -2/3 & 1/3 & 2/3 \\ 0 & 2/3 & -1/3 \\ 2/3 & 0 & 2/3 \end{bmatrix}.$$

Finally, as in Example 4 on pages 303–304,

$$\begin{aligned} R = Q^T A &= \frac{1}{3}\begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 5 & 5 \\ 0 & 2 & 1 \\ 2 & -4 & 0 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 9 & -18 & -9 \\ 0 & 9 & 9 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

5. (20 points) The linear system

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 11 \\ -x_1 + x_2 + x_3 &= 11 \\ x_1 - x_2 - x_3 &= 0 \\ x_2 + 2x_3 &= 11 \end{aligned}$$

is inconsistent. Find the normal equations that determine a least squares solution to this system. (Do not solve them.)

This linear system can be written as $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 11 \\ 11 \\ 0 \\ 11 \end{bmatrix} .$$

The normal equations are $A^T A\vec{x} = A^T\vec{b}$. We have

$$A^T A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 4 & 7 \\ 1 & 7 & 15 \end{bmatrix}$$

and

$$A^T\vec{b} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ 11 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 33 \\ 66 \end{bmatrix} .$$

Therefore the normal equations are

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 0 \\ -x_1 + 4x_2 + 7x_3 &= 33 \\ x_1 + 7x_2 + 15x_3 &= 66 . \end{aligned}$$

6. (25 points) Find a general solution to the differential equation

$$y'' - 2y' + y = 4e^t + 3t .$$

The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$. A general solution to the associated homogeneous problem is $c_1 e^t + c_2 te^t$. Using the Method of Undetermined Coefficients to find a particular solution y_p involves a trial solution of the form

$$y_p = At^2e^t + Bt + C .$$

One has

$$\begin{aligned} y_p' &= 2Ate^t + At^2e^t + B \\ y_p'' &= 2Ae^t + 4Ate^t + At^2e^t . \end{aligned}$$

Plugging this into the left-hand side of the differential equation gives

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2Ae^t + 4Ate^t + At^2e^t - 4Ate^t - 2At^2e^5 - 2B + At^2e^t + Bt + C \\ &= 2Ae^t + Bt + (C - 2B) \ . \end{aligned}$$

Setting this equal to $4e^t + 3t$ and equating coefficients gives $2A = 4$, $B = 3$, and $C - 2B = 0$; therefore $A = 2$, $B = 3$, and $C = 6$. This gives $y_p = 2t^2e^t + 3t + 6$, so a general solution to the nonhomogeneous problem is

$$y = 2t^2e^t + 3t + 6 + c_1e^t + c_2te^t \ .$$

7. (16 points) Express the system

$$\begin{aligned} x'' + 3x' + 2x + 7y &= e^t \\ y' + x' + x - y &= \cos t \\ x(0) = 3, x'(0) = 5, y(0) &= -1 \end{aligned}$$

as a matrix system in the form $\vec{x}' = A\vec{x} + \vec{f}$, $\vec{x}(0) = \vec{x}_0$. (Do not solve the system.)

With the assignments in the left-hand column and the derivatives in the right:

$$\begin{aligned} x_1 &= x \qquad & x_1' &= x' = x_2 \\ x_2 &= x' \qquad & x_2' &= x'' = e^t - 3x' - 2x - 7y = e^t - 3x_2 - 2x_1 - 7x_3 \\ x_3 &= y \qquad & x_3' &= y' = \cos t - x' - x + y = \cos t - x_2 - x_1 + x + 3 \ , \end{aligned}$$

we have

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & -7 \\ -1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^t \\ \cos t \end{bmatrix} \ .$$

8. (30 points) (a). Find a fundamental solution set for the matrix system $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}$.

The characteristic polynomial of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 = \lambda^2 - 4\lambda + 5 \ ,$$

which has roots $2 \pm i$. The eigenspace for $2+i$ is

$$\text{Nul} \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\} \ ,$$

so an eigenvector is $\begin{bmatrix}0\\1\end{bmatrix}+i\begin{bmatrix}1\\0\end{bmatrix}$. By the box on page 540, a fundamental solution set is therefore

$$e^{2t}(\cos t)\begin{bmatrix}0\\1\end{bmatrix}-e^{2t}(\sin t)\begin{bmatrix}1\\0\end{bmatrix}\;,\;e^{2t}(\sin t)\begin{bmatrix}0\\1\end{bmatrix}+e^{2t}(\cos t)\begin{bmatrix}1\\0\end{bmatrix}\;.$$

(b). Compute the Wronskian associated to this solution set.

$$\begin{vmatrix}-e^{2t}\sin t & e^{2t}\cos t\\ e^{2t}\cos t & e^{2t}\sin t\end{vmatrix}=e^{4t}\begin{vmatrix}-\sin t & \cos t\\ \cos t & \sin t\end{vmatrix}=e^{4t}(-\sin^2 t-\cos^2 t)=-e^{4t}\;.$$

9. (25 points) (a). Compute the Fourier cosine series for the function $f(x)=e^x$, $0<x<1$.

We have

$$a_0=2\int_0^1 e^x\,dx=2e^x\Big|_0^1=2e-2$$

and, for $n>0$,

$$\begin{aligned}a_n&=2\int_0^1 e^x\cos n\pi x\,dx\\&=\frac{2e^x(\cos n\pi x+n\pi\sin n\pi x)}{n^2\pi^2+1}\Bigg|_0^1\\&=\frac{2(e\cos n\pi-1)}{n^2\pi^2+1}\\&=\frac{2((-1)^n e-1)}{n^2\pi^2+1}\;.\end{aligned}$$

Therefore

$$f(x)\sim(e-1)+2\sum_{n=1}^{\infty}\frac{(-1)^n e-1}{n^2\pi^2+1}\cos n\pi x\;.$$

(b). Determine the function that this series converges to, on the interval $[-1,1]$.

It converges to $e^{|x|}$.

10. (32 points) Find a formal solution to the vibrating-string problem governed by the initial-value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 9\frac{\partial^2 u}{\partial x^2}\,, \qquad 0 < x < \pi\,, \quad t > 0\,; \\ u(0,t) = u(\pi,t) &= 0\,, \qquad t > 0\,; \\ u(x,0) &= x\,, \qquad 0 < x < \pi\,; \\ \frac{\partial u}{\partial t}(x,0) &= \sin 3x + \sin 6x\,, \qquad 0 < x < \pi\,. \end{aligned}$$

You may use memorized formulas about the wave equation for this problem (i.e., you do not have to re-derive the solution).

From the above equations, we have $\alpha = 3$ and $L = \pi$. As in Example 2 on pages 576–577,

$$\frac{\partial u}{\partial t}(x,0) = \sin 3x + \sin 6x = \sum_{n=1}^{\infty} n3b_n \sin nx\,,$$

so

$$b_3 = \frac{1}{9} \qquad \text{and} \qquad b_6 = \frac{1}{18}\,.$$

All other b_n are zero.

The values of a_n can be determined from the Fourier sine series for $f(x) = x$ given at the beginning of the exam booklet:

$$a_n = \frac{2(-1)^{n+1}}{n}\,, \qquad n = 1, 2, 3, \dots\,.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cos 3nt \sin nx + \frac{1}{9}\sin 9t \sin 3x + \frac{1}{18}\sin 18t \sin 6x\,.$$