Math 54 Sample Final Exam

In this exam, the following formulas were given:

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$$

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx\ , \qquad -\pi < x < \pi \text{ or } 0 < x < \pi$$
$$x \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos nx\ , \qquad 0 < x < \pi$$

1. (12 points) Find the inverse of the matrix $A = \begin{bmatrix} 7 & 10 & -9 \\ 1 & 2 & -3 \\ -1 & 1 & -6 \end{bmatrix}$, if it exists. Use the algorithm from the book (or from class).

2. (20 points) Let A be the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$, where x is a real number.

   (a). For which values of x is A similar to a (real) diagonal matrix? (Do not diagonalize the matrix.)

   (b). For which values of x is A orthogonally diagonalizable?

3. (20 points) Each of the following parts gives vector spaces V and W, bases $\mathcal{B}$ for V and $\mathcal{C}$ for W, and a linear transformation $T\colon V \to W$. In each case find the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$.

   (a). $V = W = \mathbb{R}^2$, $\mathcal{B} = \{(1,1),(-1,1)\}$, $\mathcal{C} = \{(1,0),(0,1)\}$, and T is counterclockwise rotation by 90 degrees.

   (b). $V = W = \operatorname{Span}\{\sin x, \cos x\} \subseteq C[0,2\pi]$, $\mathcal{B} = \mathcal{C} = \{\sin x, \cos x\}$, and T is the linear transformation taking a function to its derivative.

   (c). $V = \mathbb{R}^3$, $W = \operatorname{Span}\{(1,2,3)\} \subseteq \mathbb{R}^3$, $\mathcal{B}$ is the standard basis of $\mathbb{R}^3$, $\mathcal{C} = \{(1,2,3)\}$, and T is the projection of a vector in $\mathbb{R}^3$ to the line through $(1,2,3)$.

4. (25 points) Let A be the 4×3 matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 5 & 5 \\ 0 & 2 & 1 \\ 2 & -4 & 0 \end{bmatrix}$.

   Write $A = QR$, where Q is a matrix whose columns form an orthonormal basis for Col A and R is an upper triangular invertible matrix with positive entries on its main diagonal.

5. (20 points) The linear system

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 11 \\ -x_1 + x_2 + x_3 &= 11 \\ x_1 - x_2 - x_3 &= 0 \\ x_2 + 2x_3 &= 11 \end{aligned}$$

   is inconsistent. Find the normal equations that determine a least squares solution to this system. (Do not solve them.)

6. (25 points) Find a general solution to the differential equation

$$y'' - 2y' + y = 4e^t + 3t\ .$$

7. (16 points) Express the system

$$\begin{aligned} x'' + 3x' + 2x + 7y &= e^t \\ y' + x' + x - y &= \cos t \\ x(0) = 3, x'(0) = 5, y(0) &= -1 \end{aligned}$$

   as a matrix system in the form $\vec{x}' = A\vec{x} + \vec{f}$, $\vec{x}(0) = \vec{x}_0$. (Do not solve the system.)

8. (30 points) (a). Find a fundamental solution set for the matrix system $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}$.

   (b). Compute the Wronskian associated to this solution set.

9. (25 points) (a). Compute the Fourier cosine series for the function $f(x) = e^x$, $0 < x < 1$.

   (b). Determine the function that this series converges to, on the interval $[-1, 1]$.

10. (32 points) Find a formal solution to the vibrating-string problem governed by the initial-value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 9\frac{\partial^2 u}{\partial x^2}\ , && 0 < x < \pi\ , \quad t > 0\ ; \\ u(0,t) = u(\pi, t) &= 0\ , && t > 0\ ; \\ u(x, 0) &= x\ , && 0 < x < \pi\ ; \\ \frac{\partial u}{\partial t}(x, 0) &= \sin 3x + \sin 6x\ , && 0 < x < \pi\ . \end{aligned}$$

    You may use memorized formulas about the wave equation for this problem (i.e., you do not have to re-derive the solution).