MATH 112A – FINAL EXAM

Name: ________________________

Student ID: ________________________

Instructions: This is it, your final hurdle to freedom!!! You have 120 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your answer, but also on your work, so please write in complete sentences and explain your steps as much as you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. **You will lose 1 point if you don't fill out all the information on this page.** May your luck be fundamental!

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: ________________________________

1		5
2		25
3		20
4		5
5		10
6		10
7		10
8		15
Total		100

Date: Wednesday, December 11, 2019.

1. *(5 points, 1 point each)* Circle the correct answer or fill in the blanks. No justification required, except for (c).

   (a) The PDE $2u_{xx} + (x^2)\, u_{xy} + (e^y)\, u = \sin(x)$ is

   LINEAR   NONLINEAR

   (b) The PDE $2u_{xx} + 5u_y = 3u$ is:

   HOMOGENEOUS   INHOMOGENEOUS

   (c) The type of the second-order PDE $2u_{xx} + 3u_{xy} + u_{yy} + 6u_x + 8u_y - u = 0$ is

   ________________________ because:

   ____

   (d) Write out your favorite PDE that is neither first-order nor the Laplace/heat/wave equation

   ____

2. *(25 points)* Find a solution to the following PDE. Here $0 < x < 1$ and $0 < y < \pi$

$$\begin{cases} \left(x^2\right) u_{xx} + (x)\, u_x + u_{yy} = 0 \\ u(x,0) = 0 \quad u(x,\pi) = 0 \\ \lim_{x \to 0^+} u(x,y) = 0 \quad u(1,y) = y \end{cases}$$

Hint: At some point, you'll have to solve a strange ODE. For this ODE, guess that your solution is of the form x^α for some α (or y^α if you're dealing with Y), solve for α and then take linear combinations.

3. *(20 points)* Solve the following PDE:

$$\begin{cases} 4u_{tt} - 5u_{xt} + u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Note: Derive everything from scratch. The *only* thing you're allowed to assume is how to solve first-order PDEs.

4. *(5 points)* Find a solution of the following heat equation on the half-line (here $x > 0$)

$$\begin{cases} u_t = ku_{xx} \\ u_x(0,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Write your solution in terms of one integral. No need to write out the explicit formula for $S(x,t)$ and no need to check that your solution works.

5. *(10 points)* Find the Fourier sine series of $f(x) = \cos(x)$ on $(0, \pi)$. Simplify your expression as much as you can. Is there a contradiction when you plug in $x = 0$? Explain.

Hint:

$$\cos(A)\sin(B) = \frac{1}{2}[\sin(B + A) + \sin(B - A)]$$

6. *(10 points)* Derive Parseval's identity for the following expansion on $(0, \pi)$

$$1 = \sum_{m=0}^{\infty} A_m \cos\left(\left(\frac{2m+1}{2}\right) x\right)$$

and use it to calculate the sum

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Note: This is a series we haven't seen before, so you have to derive everything from scratch. The **only** thing you're allowed to assume is that the above cosine functions are orthogonal

7. *(10 points)* Show that the only solution of the following PDE for $0 < x < 1$ is the zero-solution.

$$\begin{cases} u_t = -u_{xxxx} - u^7 \\ u(0,t) = 0, u_x(0,t) = 0 \\ u(1,t) = 0, u_x(1,t) = 0 \\ u(x,0) = 0 \end{cases}$$

8. *(15 = 7 + 8 points)* The grand finale!!!

(a) **Definition:** If $\mathbf{F} = (F_1, \cdots, F_n)$ is a vector field in $\mathbb{R}^n$ and f is a function, then $f\,\mathbf{F} = (fF_1, \cdots, fF_n)$ (you multiply each component by f). Show that

$$div\,(f\,\mathbf{F}) = f\,\,(div\,(\mathbf{F})) + (\nabla f)\cdot\mathbf{F}$$

(b) Let D be a (connected) region in $\mathbb{R}^n$. Use (a) to solve the following Laplace equation with Neumann boundary conditions:

$$\begin{cases} \Delta u = 0 \text{ in D} \\ \dfrac{\partial u}{\partial n} = 0 \text{ on bdy D} \end{cases}$$