MOCK FINAL

Instructions: This is a mock final, designed to give you some practice for the actual midterm. Beware that the final will be harder than this one, so please also look at the study guide and the homework for a more complete study experience.

1		25
2		15
3		10
4		10
5		5
6		10
7		15
8		10
Total		100

Date: Wednesday, December 11, 2019.

1. *(25 points)* Solve the following wave equation, where $0 < x < \pi$:

$$\begin{cases} u_{tt} &= c^2 u_{xx} \\ u_x(0,t) &= 0 \\ u_x(\pi,t) &= 0 \\ u(x,0) &= x^2 \\ u_t(x,0) &= 2\cos(x) + 3\cos(2x) \end{cases}$$

2. *(15 = 10 + 5 points)*

(a) Show that Laplace's equation $u_{xx} + u_{yy} = 0$ is invariant under rotations

(b) Laplace's equation in polar coordinates becomes (you do not have to show this)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Use this to find a nonzero solution of $u_{xx} + u_{yy} = 0$

3. *(10 points)* Derive the solution of the following wave equation on the half-line (here $x > 0$)

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

4. *(10 points)* Suppose u solves (here $0 < x < 1$)

$$\begin{cases} u_t = ku_{xx} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Show

$$max\,|u| \leq \max|\phi|$$

5. *(5 points)* Find the general solution of

$$g(x)u_x + h(y)u_y = 0$$

Note: Your solution might involve integrals

6. *(10 points)* Consider the following dissipative wave equation on the whole line, where $r > 0$

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

Use energy method to show that a certain energy (which you'll have to find) decreases. Assume that any terms at $\pm\infty$ are 0

7. *(15 = 10 + 5 points)*
   (a) Find the Fourier sine series of $f(x) = x^2 + 1$ on $(0, 2)$
   (b) Sketch the graph of the function to which the above sine series converges to on $[-4, 4]$

8. *(10 = 5 + 5 points)* Suppose $u = u(x, y)$ and $v = v(x, y)$ are (real) functions and set $f = u + iv$. It is a standard fact in complex analysis that if f is (complex) differentiable, then u and v satisfy the following Cauchy-Riemann equations (don't prove it, but see Cauchy-Riemann for a proof if you're interested):

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

(a) Show that in this case, u and v both solve Laplace's equation.
(b) Let $f(z) = e^z$, where $z = x + iy$. Use (a) to find two nonzero solutions of Laplace's equation.

Sidenote: This is another way of finding solutions of Laplace's equation: Just take any differentiable f and find u and v.