

MOCK FINAL SOLUTIONS

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PROBLEM 1:

STEP 1: Separation of variables

Suppose:

$$u(x,t) = X(x) T(t) \quad (*)$$

Plug (*) into $u_{tt} = c^2 u_{xx}$

$$(X(x) T(t))_{tt} = c^2 (X(x) T(t))_{xx}$$

$$X(x) T''(t) = c^2 X''(x) T(t)$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \lambda$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \lambda \Rightarrow X''(x) = \lambda X(x)$$

$$\text{And } \frac{T''(t)}{c^2 T(t)} = \lambda \Rightarrow T''(t) = c^2 \lambda T(t)$$

STEP 2: $X(x)$ equation

So far: $X''(x) = \lambda X(x)$

Now use the boundary conditions:

$$u_x(0,t) = 0 \Rightarrow X'(0) T(t) = 0 \Rightarrow \underline{X'(0) = 0}$$

(Again, can cancel out because otherwise get 0 solution)

$$\text{Similarly } u_x(\pi,t) = 0 \Rightarrow X'(\pi) T(t) = 0 \Rightarrow \underline{X'(\pi) = 0}$$

Hence we get the ODE

$$\begin{cases} X''(x) = \lambda X(x) \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

STEP 3: 3 cases

CASE 1: $\lambda > 0$

Then $\lambda = \omega^2$ for some $\omega > 0$

Then: $X'' = \lambda X \Rightarrow X'' = \omega^2 X \Rightarrow X'' - \omega^2 X = 0$

$$\text{Aux: } r^2 - \omega^2 = 0 \Rightarrow r^2 = \omega^2 \Rightarrow r = \pm\omega$$

$$\Rightarrow X(x) = A e^{\omega x} + B e^{-\omega x} \Rightarrow X'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x}$$

But $X'(0) = A\omega - B\omega = (A - B)\omega = 0$ (since $X'(0) = 0$)

$$\Rightarrow A - B = 0 \text{ (since } \omega > 0\text{)} \Rightarrow A = B$$

$$\text{So } X(x) = A e^{\omega x} - A e^{-\omega x}$$

$$\begin{aligned}\text{But } X'(\pi) &= 0 \Rightarrow A\omega e^{\omega\pi} - A\omega e^{-\omega\pi} = 0 \\ &\Rightarrow A\omega(e^{\omega\pi} - e^{-\omega\pi}) = 0 \\ &\Rightarrow e^{\omega\pi} - e^{-\omega\pi} = 0 \\ &\Rightarrow e^{\omega\pi} = e^{-\omega\pi} \\ &\Rightarrow \omega\pi = -\omega\pi \\ &\Rightarrow \omega = 0\end{aligned}$$

But then $\lambda = \omega^2 = 0 \Rightarrow \lambda = 0$ (since we assumed $\lambda > 0$)

CASE 2: $\lambda = 0$

$$\begin{aligned}\text{Then } X'' &= 0 \Rightarrow X''(x) = 0 \\ &\Rightarrow X(x) = Ax + B\end{aligned}$$

$$\begin{aligned}X'(0) &= A = 0 \Rightarrow X(x) = 0x + B = B \\ \text{But then automatically } X'(\pi) &= 0\end{aligned}$$

So $\lambda = 0$ is a valid eigenvalue with $X(x) = B$

CASE 3: $\lambda < 0$

Then $\lambda = -\omega^2$ for some $\omega > 0$

$$X'' = \lambda X \Rightarrow X'' = -\omega^2 X \Rightarrow X'' + \omega^2 X = 0$$

$$X(x) = A \cos(\omega x) + B \sin(\omega x)$$

$$X'(x) = -A\omega \sin(\omega x) + B\omega \cos(\omega x)$$

$$X'(0) = -A\omega \sin(0) + B\omega \cos(0) = B\omega = 0$$

$\Rightarrow B = 0$ and

$$X(x) = A \cos(\omega x)$$

$$X'(\pi) = -A\omega \sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0$$

$$\Rightarrow \omega\pi = \pi m \quad (m = 1, \dots)$$

$$\Rightarrow \omega = m$$

Answer: For every $m = 1, \dots$, we have a solution,

$$X(x) = \cos(\omega x) = \cos(mx) \quad (m = 1, 2, \dots)$$

STEP 4: Now we get the equation

$$T''(t) = c^2 \lambda T(t)$$

But $\lambda = -m^2$ with $m = 0, 1, 2, \dots$ (from STEP 3)

If $m = 0$, then get $T''(t) = 0 \Rightarrow T(t) = A_0 + B_0 t$

$$u(x, t) = X(x) T(t) = (A_0 + B_0 t) \cos(0x) = A_0 + B_0 t$$

If $m = 1, 2, \dots$, then get $T''(t) = c^2 (-m^2) T(t) = -(mc)^2 T(t)$

$$\Rightarrow T(t) = A_m \cos(mct) + B_m \sin(mct)$$

$$u(x, t) = X(x) T(t) = [A_m \cos(mct) + B_m \sin(mct)] \cos(mx)$$

$$(m = 1, 2, \dots)$$

STEP 5: Linear combos:

$$u(x,t) = (A_0 + B_0 t) + \sum_{M=1}^{\infty} [A_m \cos(mx) + B_m \sin(mx)] \cos(mct)$$

STEP 6: $u(x,0) = x^2$

$$\begin{aligned} u(x,0) &= (A_0 + B_0 0) + \sum_{M=1}^{\infty} [A_m \cos(0) + B_m \sin(0)] \cos(mx) \\ &= A_0 + \sum_{M=1}^{\infty} A_m \cos(mx) \\ &= \sum_{M=0}^{\infty} A_m \cos(mx) = x^2 \end{aligned}$$

$$A_0 = \frac{\int_0^{\pi} x^2 dx}{\int_0^{\pi} 1 dx} = \frac{(\pi^3 / 3)}{\pi} = \frac{\pi^2}{3}$$

And for $m = 1, 2, \dots$

$$A_m = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(mx) dx$$

$$\begin{aligned} &+ x^2 \quad \text{cos}(mx) \\ &- 2x \quad \text{sin}(mx)/m \\ &+ 2 \quad -\text{cos}(mx)/m^2 \\ &- 0 \quad -\text{sin}(mx)/m^3 \end{aligned}$$

$$A_m = \frac{2}{\pi} \left[x^2 \sin(mx)/m + 2x \cos(mx)/m^2 - 2 \sin(mx)/m^3 \right]_0^\pi$$

$$= \frac{2}{\pi} 2\pi \cos(\pi m)/m^2 \quad (\text{all the other terms are 0})$$

$$= \frac{4}{m^2} (-1)^m$$

So $A_0 = \frac{\pi^2}{3}$ and $A_m = \frac{4}{m^2} (-1)^m$

STEP 7: $u_t(x,0) = 2 \cos(x) + 3 \cos(2x)$

$$u(x,t) = (A_0 + B_0 t) + \sum_{m=1}^{\infty} [A_m \cos(mct) + B_m \sin(mct)] \cos(mx)$$

$$u_t(x,t) = B_0 + \sum_{m=1}^{\infty} [-mc A_m \sin(mct) + mc B_m \cos(mct)] \cos(mx)$$

$$u_t(x,0) = B_0 + \sum_{m=1}^{\infty} [-mc A_m \sin(0) + mc B_m \cos(0)] \cos(mx)$$

$$= B_0 + \sum_{m=1}^{\infty} mc B_m \cos(mx)$$

$$= 2 \cos(x) + 3 \cos(2x)$$

Comparing coefficients, we find

$$B_0 = 0, cB_1 = 2, 2c B_2 = 3, mc B_m = 0 \text{ for } m = 3, \dots$$

$$\Rightarrow B_0 = 0, B_1 = 2/c, B_2 = 3/(2c), \text{ all other } B_m = 0$$

STEP 8: Conclusion

$$\begin{aligned} u(x,t) &= (A_0 + B_0 t) + \sum_{m=1}^{\infty} [A_m \cos(mct) + B_m \sin(mct)] \cos(mx) \\ &= (\pi^2/3) + \sum_{m=1}^{\infty} [4/m^2 (-1)^m \cos(mct) + B_m \sin(mct)] \cos(mx) \end{aligned}$$

With $B_1 = 2/c, B_2 = 3/(2c)$ and all other $B_m = 0$

PROBLEM 2:

(a)

Let θ be fixed and define:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (\text{Rotation matrix})$$

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases}$$

Then $u_{x'}x' + u_{y'}y' = u_{xx} + u_{yy} = 0$

Why? Use the Chen Lu!

$$u_x = \frac{\partial U}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial U}{\partial y'} \frac{\partial y'}{\partial x}$$

$$= u_{x'} \cos(\theta) + u_{y'} \sin(\theta) \quad (*)$$

$$u_{xx} = \frac{\partial U_x}{\partial x}$$

$$= \frac{\partial U_x}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial U_x}{\partial y'} \frac{\partial y'}{\partial x}$$

$$= (u_{x'} \cos(\theta) + u_{y'} \sin(\theta))_{x'} \cos(\theta) + (u_{x'} \cos(\theta) + u_{y'} \sin(\theta))_{y'} \sin(\theta)$$



$$= u_{x'x'} \cos^2(\theta) + 2 u_{x'y'} \cos(\theta)\sin(\theta) + u_{y'y'} \sin^2(\theta)$$

Similarly:

$$u_{yy} = u_{x'x'} \sin^2(\theta) - 2 u_{x'y'} \cos(\theta)\sin(\theta) + u_{y'y'} \cos^2(\theta)$$

Therefore:

$$u_{xx} + u_{yy} = u_{x'x'} (\cos^2(\theta) + \sin^2(\theta)) + 2 u_{x'y'} \cos(\theta)\sin(\theta)$$

$$-2 u_{x'y'} \cos(\theta)\sin(\theta) + u_{y'y'} (\sin^2(\theta) + \cos^2(\theta))$$

$$= u_{x'x'} + u_{y'y'}$$

$$\text{Hence } u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$$

(b)

Laplace's equation in polar coordinates is

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0 \quad (*)$$

If u is radial, then $u_\theta = 0$, so $u_{\theta\theta} = 0$, and (*) becomes:

$$u_{rr} + \frac{u_r}{r} = 0$$

$$u_{rr} = -\frac{u_r}{r}$$

$$\frac{u_{rr}}{u_r} = -\frac{1}{r}$$

$$(\ln|u_r|)' = -1/r$$

$$\Rightarrow \ln|u_r| = -\ln(r) + C$$

$$\Rightarrow |u_r| = e^{-\ln(r) + C} = \frac{e^C}{e^{\ln(r)}}$$

$$\Rightarrow u_r = \frac{+/- e^C}{r} = \frac{C}{r}$$

$$\Rightarrow u = C \ln(r) + C'$$

$$u(x,y) = C \ln(\sqrt{x^2 + y^2}) + C' \text{ solves } u_{xx} + u_{yy} = 0$$

PROBLEM 3:

STEP 1: Oddify both ϕ and ψ

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ -\phi(-x) & \text{if } x < 0 \end{cases}$$

$$\psi_{\text{odd}}(x) = \begin{cases} \psi(x) & \text{if } x > 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}$$

STEP 2: Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi_{\text{odd}}(x) \\ u_t(x,0) = \psi_{\text{odd}}(x) \end{cases}$$

\Rightarrow D'Alembert:

$$u(x,t) = 1/2 (\phi_{\text{odd}}(x-ct) + \phi_{\text{odd}}(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) ds$$

STEP 3: Write in terms of ϕ and ψ

CASE 1: $x-ct > 0$

Then $\phi_{\text{odd}}(x-ct) = \phi(x-ct)$ (and $\phi_{\text{odd}}(x+ct) = \phi(x+ct)$) and $\psi_{\text{odd}}(s) = \psi(s)$ on $[x-ct, x+ct]$ (since $x-ct > 0$), so

$$u(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{(2c)} \int_{x-ct}^{x+ct} \psi(s) ds$$

CASE 2: $x-ct < 0$

Then $\phi_{\text{odd}}(x-ct) = -\phi(-(x-ct)) = -\phi(ct-x)$
 (and $\phi_{\text{odd}}(x+ct) = \phi(x+ct) = \phi(ct+x)$)

And

$$\begin{aligned} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) ds &= \int_{x-ct}^0 \psi_{\text{odd}}(s) ds + \int_0^{x+ct} \psi_{\text{odd}}(s) ds \\ &= \int_{x-ct}^0 -\psi(-s) ds + \int_0^{x+ct} \psi(s) ds \\ p = -s &\quad \downarrow \\ dp = -ds &= \int_{-x+ct}^0 \psi(p) dp + \int_0^{x+ct} \psi(s) ds \\ &= \int_{ct-x}^{ct+x} \psi(s) ds \end{aligned}$$

$$u(x,t) = \frac{1}{2} (\phi(ct+x) - \phi(ct-x)) + \frac{1}{(2c)} \int_{ct-x}^{ct+x} \psi(s) ds$$

PROBLEM 4:

Note: This is similar to the proof of stability from Lecture 12

Let $M = \max |\phi|$

1) By the maximum principle,

$$\begin{aligned}\max u &= \text{The larger one of } \max u(0,t), \max u(1,t), \max u(x,0) \\ &= \text{The larger one of } 0, 0, \max \phi\end{aligned}$$

$$\text{But } 0 \leq M, 0 \leq M, \text{ and } \max \phi \leq \max |\phi| = M$$

So in any case $\max u \leq M$

In particular, $u(x,t) \leq M$ for all (x,t)

2) Similarly, by the minimum principle

$$\begin{aligned}\min u &= \text{The smaller one of } \min u(0,t), \min u(1,t), \min u(x,0) \\ &= \text{The smaller one of } 0, 0, \min \phi\end{aligned}$$

$$\text{But } 0 \geq -M, 0 \geq -M, \min f \geq \min -|\phi| = -\max |\phi| = -M$$

(Here we used $z \geq -|z|$ for every z , and $\min -z = -\max z$)

Hence $\min u \geq -M$

And so in particular, $u(x,t) \geq -M$ for all (x,t)

3) Combining 1) and 2), we get

$$-M \leq u(x,t) \leq M \text{ for all } (x,t)$$

$$\Rightarrow |u(x,t)| \leq M$$

And taking the max over all (x,t) , we get

$$\max |u| \leq M$$

So $\max |u| \leq \max |\phi|$ (since $M = \max |\phi|$)

PROBLEM 5:

$$g(x) u_x + h(y) u_y = 0$$

$$\frac{dy}{dx} = \text{Slope} = \frac{h(y)}{g(x)}$$

$$\Rightarrow g(x) dy = h(y) dx$$

$$\Rightarrow \frac{dy}{h(y)} = \frac{dx}{g(x)}$$

$$\Rightarrow \int \frac{dy}{h(y)} = \int \frac{dx}{g(x)}$$

$$\Rightarrow \tilde{H}(y) = \tilde{G}(x) + C$$

where \tilde{H} is an antiderivative of $1/h$
 \tilde{G} is an antiderivative of $1/g$

$$\Rightarrow C = \tilde{H}(y) - \tilde{G}(x)$$

Solution: $u(x,y) = f(\tilde{H}(y) - \tilde{G}(x))$ where f is arbitrary

PROBLEM 6:

Multiply $u_{tt} - c^2 u_{xx} + r u_t = 0$ by u_t

$$u_{tt} u_t - c^2 u_{xx} u_t + r u_t u_t = 0$$

Integrate with respect to x

$$\int_{-\infty}^{\infty} u_{tt} u_t dx - \int_{-\infty}^{\infty} c^2 u_{xx} u_t dx + \int_{-\infty}^{\infty} r u_t u_t dx = 0$$

$$A = \int_{-\infty}^{\infty} 1/2 d/dt (u_t)^2 dx = d/dt \left(1/2 \int_{-\infty}^{\infty} (u_t)^2 dx \right)$$

For B, integrate by parts with respect to x (assuming no terms at infinity, because they go to 0 by assumption)

$$\begin{aligned} B &= +c^2 \int_{-\infty}^{\infty} u_x u_{xt} dx = c^2 \int_{-\infty}^{\infty} 1/2 d/dt (u_x)^2 dx \\ &= d/dt 1/2 c^2 \int_{-\infty}^{\infty} (u_x)^2 dx \end{aligned}$$

Finally,

$$C = r \int_{-\infty}^{\infty} (u_t)^2 dx \geq 0$$

Combining everything, we get

$-\infty$
Combining everything, we get

$$\frac{d}{dt} \left[\frac{1}{2} \int_{-\infty}^{\infty} (u_t)^2 dx + \frac{1}{2} c^2 \int_{-\infty}^{\infty} (u_x)^2 dx \right] = -C \leq 0$$

$$\frac{d}{dt} \left[\frac{1}{2} \int_{-\infty}^{\infty} (u_t)^2 + c^2 (u_x)^2 dx \right] \leq 0$$

$$\frac{d}{dt} E(t) \leq 0$$

$$E'(t) \leq 0$$

Hence the energy $E(t)$ is decreasing, where

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u_t)^2 + c^2 (u_x)^2 dx$$

Note: The answer without $1/2$ is also acceptable

PROBLEM 7:

(a)

$$x^2 + 1 = \sum_{m=1}^{\infty} B_m \sin(\pi mx/2)$$

$$A_m = \frac{2}{2} \int_0^2 (x^2 + 1) \sin(\pi mx/2) dx$$

$$= \int_0^2 (x^2 + 1) \sin(\pi mx/2) dx$$

$$\begin{array}{rcl}
 + & x^2 + 1 & \xrightarrow{\quad} \sin(\pi m x / 2) \\
 - & 2x & \xleftarrow{\quad} -\cos(\pi m x / 2) (2/\pi m) \\
 + & 2 & \xleftarrow{\quad} -\sin(\pi m x / 2) (2/\pi m)^2 \\
 - & 0 & \xleftarrow{\quad} \cos(\pi m x / 2) (2/\pi m)^3
 \end{array}$$

$$= \left[-(x^2+1) \cos(\pi m x / 2) (2/\pi m) + 2x \sin(\pi m x / 2) (2/\pi m)^2 \right]_0^2 + 2 \cos(\pi m x / 2) (2/\pi m)^3$$

$$= -(2^2+1) \cos(\pi m) (2/\pi m) + 2 \cos(\pi m) (2/\pi m)^3 + \cos(0) (2/\pi m) - 2 \cos(0) (2/\pi m)^3$$

$$= (10(-1)^{m+1} + 2)/(\pi m) + 16 ((-1)^m - 1)/(\pi m)^3$$

(b) Basically oddify and periodify f , and apply the rules from lecture



PROBLEM 8:

(a)

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0$$

So u and v are harmonic

(b)

$$\begin{aligned} f(z) &= e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y)) \\ &= e^x \cos(y) + i e^x \sin(y) \\ &= u + i v \end{aligned}$$

So by (a), we get

$$u(x,y) = e^x \cos(y) \text{ and } v(x,y) = e^x \sin(y)$$

both solve Laplace's equation!